# Inverse Numerator Relationship Matrix

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16 November 2018

# Structure of $A^{-1}$

- Look at a simple example of $A$ and $A^{-1}$

Table 1: Pedigree Used To Compute Inverse Numerator Relationship Matrix

| Calf | Sire | Dam |
|---:|---:|---:|
| 1 | NA | NA |
| 2 | NA | NA |
| 3 | NA | NA |
| 4 | 1 | 2 |
| 5 | 3 | 2 |

# Numerator Relationship Matrix $A$

$$A = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.5000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.5000 & 0.5000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.5000 \\ 0.5000 & 0.5000 & 0.0000 & 1.0000 & 0.2500 \\ 0.0000 & 0.5000 & 0.5000 & 0.2500 & 1.0000 \end{bmatrix} \quad (1)$$

# Inverse Numerator Relationship Matrix $A^{-1}$

$$A^{-1} = \begin{bmatrix} 1.5000 & 0.5000 & 0.0000 & -1.0000 & 0.0000 \\ 0.5000 & 2.0000 & 0.5000 & -1.0000 & -1.0000 \\ 0.0000 & 0.5000 & 1.5000 & 0.0000 & -1.0000 \\ -1.0000 & -1.0000 & 0.0000 & 2.0000 & 0.0000 \\ 0.0000 & -1.0000 & -1.0000 & 0.0000 & 2.0000 \end{bmatrix}$$

# Conclusions

- $A^{-1}$ has simpler structure than $A$ itself
- Non-zero elements only at positions of parent-progeny and parent-mate positions
- Parent-mate positions are positive, parent-progeny are negative

# Henderson's Rules

- Based on LDL-decomposition of $A$

$$A = L * D * L^T$$

| | | |
|---|---|---|
| where | $L$ | Lower triangular matrix |
| | $D$ | Diagonal matrix |

- Why?
  - matrices $L$ and $D$ can be inverted directly, we 'll see how ...
  - construct $A^{-1} = (L^T)^{-1} * D^{-1} * L^{-1}$

# Example

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 1.0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

$\rightarrow$ Verify that $A = L * D * L^T$

# Decomposition of True Breeding Value

- True breeding value ($a_i$) of animal $i$

$$a_i = \frac{1}{2}a_s + \frac{1}{2}a_d + m_i$$

- Do that for all animals in pedigree

# Decomposition for Example

$$a_1 = m_1$$
$$a_2 = m_2$$
$$a_3 = m_3$$
$$a_4 = \frac{1}{2}a_1 + \frac{1}{2}a_2 + m_4$$
$$a_5 = \frac{1}{2}a_3 + \frac{1}{2}a_2 + m_5$$

# Matrix Vector Notation

- Define vectors $a$ and $m$ as
- Coefficients of $a_s$ and $a_d$ into matrix $P$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix}, P = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 0.0 \end{bmatrix}$$

- Result: Decomposition of true breeding values

$$a = P \cdot a + m$$

# Decomposition of Variance

- Analogous decomposition of $var(a_i)$

$$
\begin{aligned}
var(a_i) &= var(1/2a_s + 1/2a_d + m_i) \\
&= var(1/2a_s) + var(1/2a_d) + \frac{1}{2} * cov(a_s, a_d) + var(m_i) \\
&= 1/4var(a_s) + 1/4var(a_d) + \frac{1}{2} * cov(a_s, a_d) + var(m_i)
\end{aligned}
$$

- From the definition of $A$

$$
\begin{aligned}
var(a_i) &= (1 + F_i)\sigma_a^2 \\
var(a_s) &= (1 + F_s)\sigma_a^2 \\
var(a_d) &= (1 + F_d)\sigma_a^2 \\
cov(a_s, a_d) &= (A)_{sd}\sigma_a^2 = 2F_i\sigma_a^2
\end{aligned}
$$

# Variance of Mendelian Sampling Terms

- What is $var(m_i)$?
- Solve equation for $var(a_i)$ for $var(m_i)$

$$var(m_i) = var(a_i) - 1/4 var(a_s) - 1/4 var(a_d) - 2 * cov(a_s, a_d)$$

- Insert definitions from $A$

$$\begin{aligned} var(m_i) &= (1 + F_i)\sigma_a^2 - 1/4(1 + F_s)\sigma_a^2 - 1/4(1 + F_d)\sigma_a^2 - \frac{1}{2} * 2 * F_i \sigma_a^2 \\ &= \left(\frac{1}{2} - \frac{1}{4}(F_s + F_d)\right)\sigma_a^2 \end{aligned}$$

- True, for both parents $s$ and $d$ of animal $i$ are known

# Unknown Parents

- Only parent $s$ of animal $i$ is known

$$\begin{aligned} a_i &= \frac{1}{2} a_s + m_i \\ var(m_i) &= \left(1 - \frac{1}{4}(1 + F_s)\right) \sigma_a^2 \\ &= \left(\frac{3}{4} - \frac{1}{4} F_s\right) \sigma_a^2 \end{aligned}$$

- Both parents are unknown

$$\begin{aligned} a_i &= m_i \\ var(m_i) &= \sigma_a^2 \end{aligned}$$

# Recursive Decomposition

- True breeding values of $s$ and $d$ can be decomposed into

$$a_s = \frac{1}{2}a_{ss} + \frac{1}{2}a_{ds} + m_s$$
$$a_d = \frac{1}{2}a_{sd} + \frac{1}{2}a_{dd} + m_d$$

where

| | | |
|---|---|---|
| where | $ss$ | sire of $s$ |
| | $ds$ | dam of $s$ |
| | $sd$ | sire of $d$ |
| | $dd$ | dam of $d$ |

# Example

- Add animal 6 with parents 4 and 5 to our example pedigree

| Calf | Sire | Dam |
|---|---|---|
| 1 | NA | NA |
| 2 | NA | NA |
| 3 | NA | NA |
| 4 | 1 | 2 |
| 5 | 3 | 2 |
| 6 | 4 | 5 |

# First Step Of Decomposition

$$a_1 = m_1$$
$$a_2 = m_2$$
$$a_3 = m_3$$
$$a_4 = \frac{1}{2}a_1 + \frac{1}{2}a_2 + m_4$$
$$a_5 = \frac{1}{2}a_3 + \frac{1}{2}a_2 + m_5$$
$$a_6 = \frac{1}{2}a_4 + \frac{1}{2}a_5 + m_6$$

# Decompose Parents

$$
\begin{aligned}
a_1 &= m_1 \\
a_2 &= m_2 \\
a_3 &= m_3 \\
a_4 &= \frac{1}{2} m_1 + \frac{1}{2} m_2 + m_4 \\
a_5 &= \frac{1}{2} m_3 + \frac{1}{2} m_2 + m_5 \\
a_6 &= \frac{1}{2} \left( \frac{1}{2}(a_1 + a_2) + m_4 \right) + \frac{1}{2} \left( \frac{1}{2}(a_3 + a_2) + m_5 \right) + m_6 \\
&= \frac{1}{4}(a_1 + a_2) + \frac{1}{2} m_4 + \frac{1}{4}(a_3 + a_2) + \frac{1}{2} m_5 + m_6
\end{aligned}
$$

## Decompose Grand Parents

- Only animal 6 has true breeding values for grand parents

$$
\begin{aligned}
a_6 &= \frac{1}{4}(a_1 + a_2) + \frac{1}{2}m_4 + \frac{1}{4}(a_3 + a_2) + \frac{1}{2}m_5 + m_6 \\
&= \frac{1}{4}m_1 + \frac{1}{4}m_2 + \frac{1}{4}m_3 + \frac{1}{4}m_2 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6 \\
&= \frac{1}{4}m_1 + \frac{1}{2}m_2 + \frac{1}{4}m_3 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6
\end{aligned}
$$

# Summary

$$
\begin{aligned}
a_1 &= m_1 \\
a_2 &= m_2 \\
a_3 &= m_3 \\
a_4 &= \frac{1}{2}m_1 + \frac{1}{2}m_2 + m_4 \\
a_5 &= \frac{1}{2}m_3 + \frac{1}{2}m_2 + m_5 \\
a_6 &= \frac{1}{4}m_1 + \frac{1}{2}m_2 + \frac{1}{4}m_3 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6
\end{aligned}
$$

# Matrix-Vector Notation

- Use vectors $a$ and $m$ again

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}, \ m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}, \ L = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.50 & 0.50 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.50 & 0.50 & 0.00 & 1.00 & 0.00 \\ 0.25 & 0.50 & 0.25 & 0.50 & 0.50 & 1.00 \end{bmatrix}$$

- Result of recursive decomposition of $a_i$

$$a = L \cdot m$$

# Variance From Recursive Decomposition

$$
\begin{aligned}
var(a) &= var(L \cdot m) \\
&= L \cdot var(m) \cdot L^T
\end{aligned}
$$

where $var(m)$ is the variance-covariance matrix of all components in vector $m$.

- covariances of components $m_i$, $cov(m_i, m_j) = 0$ for $i \neq j$
- $var(m_i)$ computed as shown before

# Result

- variance-covariance matrix $var(m)$ can be written as $D * \sigma_a^2$ where $D$ is diagnoal

$$
\begin{aligned}
\rightarrow var(a) &= L \cdot var(m) \cdot L^T \\
&= L \cdot D * \sigma_a^2 \cdot L^T \\
&= L \cdot D \cdot L^T * \sigma_a^2 \\
&= A\sigma_a^2
\end{aligned}
$$

$$\rightarrow A = L \cdot D \cdot L^T$$