Inverse Numerator Relationship Matrix and Inbreeding

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Inverse of A Based on L and D

- Matrix A was decomposed into $A = L \cdot D \cdot L^T$
- Get A^{-1} as $A^{-1} = (L^T)^{-1} D^{-1} L^{-1}$
- D^{-1} is diagonal again with elements

$$(D^{-1})_{ii} = 1/(D)_{ii}$$

Inverse of L

- Compute m based on the two decompositions of a

$$a = P \cdot a + m \quad \text{and} \quad a = L \cdot m$$

- Solve both for m and set them equal

$$m = a - P \cdot a = (I - P) \cdot a \quad \text{and} \quad m = L^{-1} \cdot a$$

$$(I - P) \cdot a = L^{-1} \cdot a$$

and

$$L^{-1} = I - P$$

Example

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2

Matrix D^{-1}

- Because D is diagonal

$$D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

- We get D^{-1} as

$$D^{-1} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

Matrix L^{-1}

- Use $L^{-1} = I - P$
- Matrix P from simple decomposition

$$P = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 0.0 \end{bmatrix}$$

- Therefore

$$L^{-1} = I - P = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.5 & -0.5 & 0.0 & 1.0 \end{bmatrix}$$

Decomposition of A^{-1} I

$$A^{-1} = (L^{-1})^T \cdot D^{-1} \cdot L^{-1}$$

$$(L^{-1})^T \cdot D^{-1}$$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 & -0.5 & -0.5 \\ 0.0 & 0.0 & 1.0 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \cdot \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

Decomposition of A^{-1} II

$$A^{-1} = (L^{-1})^T \cdot D^{-1} \cdot L^{-1}$$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix} \cdot \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.5 & -0.5 & 0.0 & 1.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & 0.5 & 0.0 & -1.0 & 0.0 \\ 0.5 & 2.0 & 0.5 & -1.0 & -1.0 \\ 0.0 & 0.5 & 1.5 & 0.0 & -1.0 \\ -1.0 & -1.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & -1.0 & -1.0 & 0.0 & 2.0 \end{bmatrix}$$

Henderson's Rules

- Both Parents Known
  - add 2 to the diagonal-element (i, i)
  - add -1 to off-diagonal elements (s, i), (i, s), (d, i) and (i, d)
  - add $\frac{1}{2}$ to elements (s, s), (d, d), (s, d), (d, s)
- Only One Parent Known
  - add $\frac{4}{3}$ to diagonal-element (i, i)
  - add $-\frac{2}{3}$ to off-diagonal elements (s, i), (i, s)
  - add $\frac{1}{3}$ to element (s, s)
- Both Parents Unknown
  - add 1 to diagonal-element (i, i)
- Valid without inbreeding

Inbreeding

- Elements in matrix D depend on coefficients of inbreeding
- Recap: From the simple decomposition of a, we derived

$$
\begin{aligned}
var(m_i) &= \left(\frac{1}{2} - \frac{1}{4}(F_s + F_d)\right) \sigma_a^2 \\
&= \left(\frac{1}{2} - \frac{1}{4}(A_{ss} - 1 + A_{dd} - 1)\right) \sigma_a^2 \\
&= \left(1 - \frac{1}{4}(A_{ss} + A_{dd})\right) \sigma_a^2 \\
&= (D)_{ii}\sigma_a^2
\end{aligned}
$$

$$
\rightarrow (D)_{ii} = \left(\frac{1}{2} - \frac{1}{4}(F_s + F_d)\right) = \left(1 - \frac{1}{4}(A_{ss} + A_{dd})\right)
$$

Computation of Coefficients of Inbreeding

- Observation: Coefficients of inbreeding F_s and F_d can be read from $(A)_{ss}$ and $(A)_{dd}$ of A
- Cannot setup A to just get inbreeding coefficients
- More efficient method required
- **Cholesky** decomposition of A

$$A = R \cdot R^T$$

where R is a lower triangular matrix

Hint: Function `chol(A)` in R gives matrix R^T

Cholesky Decomposition

- Diagonal elements $(A)_{ii}$ of A are the sum of the squared elements of one row of R

$$(A)_{ii} = \sum_{j=1}^{i} (R)_{ij}^2$$

- Example

$$\begin{bmatrix} (A)11 & (A)12 & (A)13 \\ (A)21 & (A)22 & (A)23 \\ (A)31 & (A)32 & (A)33 \end{bmatrix} = \begin{bmatrix} (R)11 & 0 & 0 \\ (R)21 & (R)22 & 0 \\ (R)31 & (R)32 & (R)33 \end{bmatrix} \cdot \begin{bmatrix} (R)11 & (R)21 & (R)31 \\ 0 & (R)22 & (R)32 \\ 0 & 0 & (R)33 \end{bmatrix}$$

Recursive Computation of R

- Let us write the matrix R as a product of two matrices L and S:

$$R = L \cdot S$$

where L is the same matrix as in the LDL-decompositon and S is a diagonal matrix.

- Compute A as

$$A = R \cdot R^T = L \cdot S \cdot S \cdot L^T = L \cdot D \cdot L^T$$

- Hence

$$D = S \cdot S \quad \rightarrow \quad (S)_{ii} = \sqrt{(D)_{ii}}$$

Example

$$\begin{bmatrix} (R)11 & 0 & 0 \\ (R)21 & (R)22 & 0 \\ (R)31 & (R)32 & (R)33 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (L)21 & 1 & 0 \\ (L)31 & (L)32 & 1 \end{bmatrix} \cdot \begin{bmatrix} (S)11 & 0 & 0 \\ 0 & (S)22 & 0 \\ 0 & 0 & (S)33 \end{bmatrix}$$

- Diagnoal elements $(R)_{ii} = (S)_{ii}$
- Because $(S)_{ii} = \sqrt{(D)_{ii}}$, if parents s and d are known diagonal elements $(R)_{ii}$ of matrix R can be computed as

$$(R)_{ii} = (S)_{ii} = \sqrt{(D)_{ii}} = \sqrt{\left(1 - \frac{1}{4}(A_{ss} + A_{dd})\right)}$$

- A_{ss} and A_{dd} are
  - 0 if s and d are unknown (NA) or
  - have been computed before

Recap matrix D

- Both parents s and d of animal i are known

$$(D)_{ii} = \frac{1}{2} - \frac{1}{4}(F_s + F_d) = \frac{1}{2} - \frac{1}{4}((A)_{ss} - 1 + (A)_{dd} - 1) = 1 - \frac{1}{4}((A)_{ss} + (A)_{dd})$$

- Parent s of animal i is known

$$(D)_{ii} = \frac{3}{4} - \frac{1}{4}F_s = \frac{3}{4} - \frac{1}{4}((A)_{ss} - 1) = 1 - \frac{1}{4}(A)_{ss}$$

- Both parents unknown

$$(D)_{ii} = 1$$

Offdiagonal Elements of R

- Offdiagnoal elements $(R)_{ij}$ of R are computed as

$$(R)_{ij} = (L)_{ij} * (S)_{jj}$$

- Use property of L: $L_{ij} = \frac{1}{2}((L)_{sj} + (L)_{dj})$ if s and d are parents of i

$$\begin{aligned}
(R)_{ij} &= (L)_{ij} * (S)_{jj} \\
&= \frac{1}{2}\left[(L)_{sj} + (L)_{dj}\right] * (S)_{jj} \\
&= \frac{1}{2}\left[(L)_{sj} * (S)_{jj} + (L)_{dj} * (S)_{jj}\right] \\
&= \frac{1}{2}\left[(R)_{sj} + (R)_{dj}\right]
\end{aligned}$$

Example Pedigree

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2
6	4	5

Computations

- Compute diagonal elements $(A)_{ii}$ of A to get F_i
- Prerequisite: Pedigree sorted such that parents before progeny
- Start with $(A)_{11}$

$$(A)_{11} = (R)_{11}^2 = (D)_{11} = 1$$

- $(A)_{22} = (R)_{21}^2 + (R)_{22}^2 = 0 + 1 = 1$
- $(A)_{33} = (R)_{31}^2 + (R)_{32}^2 + (R)_{33}^2 = 0 + 0 + 1 = 1$

Animals With Known Parents

$$
\begin{aligned}
(A)_{44} &= (R)_{41}^2 + (R)_{42}^2 + (R)_{43}^2 + (R)_{44}^2 \\
&= (\frac{1}{2}(R_{11} + R_{21}))^2 + (\frac{1}{2}(R_{12} + R_{22}))^2 + (\frac{1}{2}(R_{13} + R_{23}))^2 \\
&+ \left(1 - \frac{1}{4}(A_{11} + A_{22})\right) \\
&= \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1
\end{aligned}
$$

- $(A)_{55}$
- $(A)_{66}$