

Numerator Relationship Matrix

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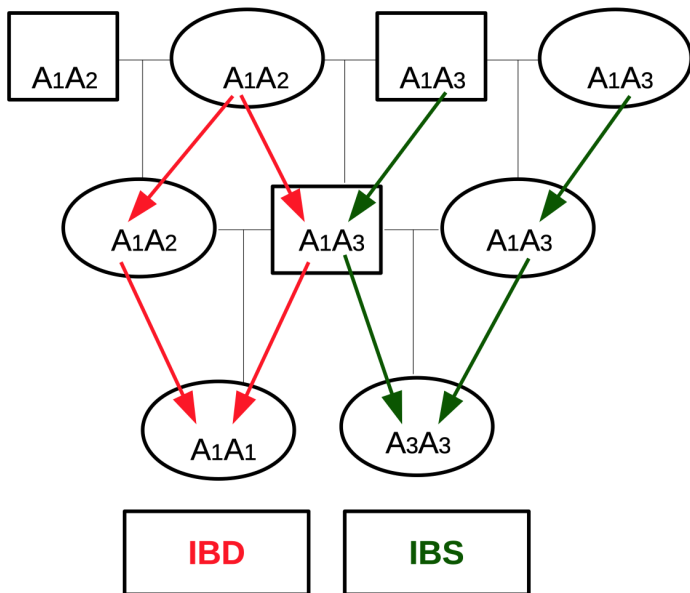
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Similarity Between Individuals

At the genetic level there are two different kinds of similarity

1. Identity by descent (IBD)
2. Identity by state

IBD versus IBS



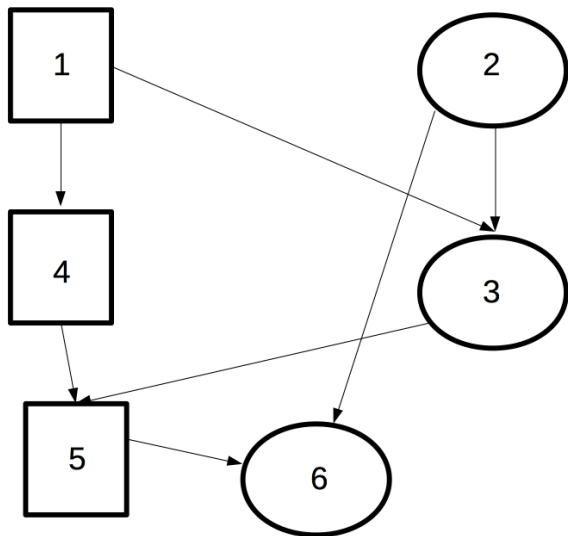
Numerator Relationship Matrix

- ▶ probability of IBD alleles in two individuals: **coancestry** or **coefficient of kinship**
- ▶ additive genetic relationship between two individuals is twice their coancestry
- ▶ matrix containing all additive genetic relationships in a population is called **numerator relationship matrix** (A)
- ▶ A is symmetric and contains on
 - ▶ diagonal: $(A)_{ii} = (1 + F_i)$
 - ▶ off-diagonal: $(A)_{ij} = \text{cov}(u_i, u_j) / \sigma_u^2$ (with $i \neq j$)

Recursive Computation of A

- ▶ If both parents s and d of animal i are known then
 - ▶ the diagonal element $(A)_{ii}$ corresponds to:
 $(A)_{ii} = 1 + F_i = 1 + \frac{1}{2}(A)_{sd}$ and
 - ▶ the offdiagonal element $(A)_{ji}$ is computed as:
 $(A)_{ji} = \frac{1}{2}((A)_{js} + (A)_{jd})$
 - ▶ because A is symmetric $(A)_{ji} = (A)_{ij}$
- ▶ If only one parent s is known and assumed unrelated to the mate
 - ▶ $(A)_{ii} = 1$
 - ▶ $(A)_{ij} = (A)_{ji} = \frac{1}{2}(A)_{js}$
- ▶ If both parents are unknown
 - ▶ $(A)_{ii} = 1$
 - ▶ $(A)_{ij} = (A)_{ji} = 0$

Example



Tabular Representation of Pedigree

Table 1: Example Pedigree To Compute Additive Genetic Relationship Matrix

Calf	Sire	Dam
3	1	2
4	1	NA
5	4	3
6	5	2

Stepwise Computation of A

- ▶ Start by extending pedigree with animals that do not have parents
- ▶ Order animals, such that parents before progeny

Animal	Sire	Dam
1	NA	NA
2	NA	NA
3	1	2
4	1	NA
5	4	3
6	5	2

First Diagonal Element

- ▶ Compute first element $(A)_{11} = 1 + F_1$
- ▶ Animal 1 has both parents unknown $\rightarrow F_1 = 0$

$$A = \begin{bmatrix} 1.00 & & \\ & & \\ & & \end{bmatrix}$$

Off-diagonal Elements

- ▶ Assume animal i has parents s and d
- ▶ $(A)_{ji} = \frac{1}{2}((A)_{js} + (A)_{jd})$

First Row of A

$$A = \begin{bmatrix} 1.00 & 0.00 & 0.50 & 0.50 & 0.50 & 0.25 \end{bmatrix}$$

Use Symmetry of A

- ▶ Copy first row into first column

$$A = \begin{bmatrix} 1.00 & 0.00 & 0.50 & 0.50 & 0.50 & 0.25 \\ 0.00 & & & & & \\ 0.50 & & & & & \\ 0.50 & & & & & \\ 0.50 & & & & & \\ 0.25 & & & & & \end{bmatrix}$$

Remaining Elements of A

- ▶ Continue with rows and columns 2 to 6 using the same recipe

Final Result

$$A = \begin{bmatrix} 1.0000 & 0.0000 & 0.5000 & 0.5000 & 0.5000 & 0.2500 \\ 0.0000 & 1.0000 & 0.5000 & 0.0000 & 0.2500 & 0.6250 \\ 0.5000 & 0.5000 & 1.0000 & 0.2500 & 0.6250 & 0.5625 \\ 0.5000 & 0.0000 & 0.2500 & 1.0000 & 0.6250 & 0.3125 \\ 0.5000 & 0.2500 & 0.6250 & 0.6250 & 1.1250 & 0.6875 \\ 0.2500 & 0.6250 & 0.5625 & 0.3125 & 0.6875 & 1.1250 \end{bmatrix}$$

The Inverse Numerator Relationship Matrix

- ▶ Recap: Henderson's mixed model equations depend on four matrices
 1. Design matrix X for the fixed effects
 2. Design matrix Z for the random effects
 3. The inverse variance-covariance matrix R^{-1} for the residuals e and
 4. The inverse variance-covariance matrix G^{-1} for the random breeding values a .

Animal Model

- ▶ Breeding values of all individuals as random effects
- ▶ Variance-Covariance matrix G corresponds to variance-covariance matrix of breeding values

$$G = A * \sigma_u^2$$

- ▶ We need: G^{-1}

$$G^{-1} = A^{-1} * \frac{1}{\sigma_u^2}$$

Need For Efficient Computation of A^{-1}

- ▶ In practical livestock breeding evaluations A is very large
- ▶ Dimensions of A can be $10^7 \times 10^7$
- ▶ Explicit general inversion not possible
- ▶ Special structure of A^{-1} leads to efficient computation