

# Livestock Breeding and Genomics - Solution 6

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## Problem 1: Numerator Relationship Matrix

Construct the numerator relationship matrix  $A$  for the following pedigree and verify the result using the function `getA()` from package `pedigreemm`.

Table 1: Pedigree For Constructing Numerator Relationship Matrix

Animal	Sire	Dam
5	1	2
6	1	3
7	4	5
8	4	5
9	4	6
10	4	6

## Solution

The numerator relationship is constructed using the following step-wise procedure. The following rules are used to compute the single elements.

- **Case 1:** If both parents  $s$  and  $d$  of animal  $i$  are known then
  - the diagonal element  $(A)_{ii}$  corresponds to:  $(A)_{ii} = 1 + F_i = 1 + \frac{1}{2}(A)_{sd}$  and
  - the offdiagonal element  $(A)_{ji}$  is computed as:  $(A)_{ji} = \frac{1}{2}((A)_{js} + (A)_{jd})$
  - because  $A$  is symmetric  $(A)_{ji} = (A)_{ij}$
- **Case 2:** If only one parent  $s$  is known and assumed unrelated to the mate
  - $(A)_{ii} = 1$
  - $(A)_{ij} = (A)_{ji} = \frac{1}{2}(A)_{js}$
- **Case 3:** If both parents are unknown
  - $(A)_{ii} = 1$
  - $(A)_{ij} = (A)_{ji} = 0$

## Step 1

First, we extend the pedigree given in Table 1. All animals without parents are added at the top of the pedigree. This results in the matrix shown in Table 2.

Table 2: Extended Pedigree

Animal	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	NA	NA
5	1	2
6	1	3

7	4	5
8	4	5
9	4	6
10	4	6

Because the pedigree in Table 2 is already ordered such that parents are before offspring, we can directly go to the next step.

### Step 2

We start with an empty numerator relationship matrix  $A$ . The matrix  $A$  has dimensions  $10 \times 10$

$$A = \left[ \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right]$$

### Step 3

The single elements of  $A$  are computed according to the rules listed above.

The computation is started with animal 1. The first element is always the diagonal-element that corresponds to animal that we are currently looking at. For animal 1 the diagonal element is  $(A)_{11}$ . Because animal 1 has not parents, we are in case 3 for the diagonal element. If an animal has unknown parents, it also means that the animals's inbreeding coefficient  $F_i$  is 0. Hence

$$(A)_{11} = 1$$

Now we have the first element of our numerator relationship matrix.

$$A = \left[ \begin{array}{c} 1.00 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right]$$

### Step 4

The next elements that need to be computed are the off-diagonal element on row 1. Elements  $A_{12}$ ,  $A_{13}$  and  $A_{14}$  correspond to the additive genetic relationship between animal 1 and animals 2, 3 and 4. Because animals 2, 3 and 4 all have unknown parents, we are for all three elements in case 3, hence we can state



## Step 6

Continue the same way with rows 2 to 10

### Verification

We first have to specify the pedigree, before being able to get the numerator relationship matrix

```
n_nr_animals <- 10
suppressPackageStartupMessages( library(pedigreeemm) )
ped <- pedigree(sire = c(NA,NA,NA,NA,1,1,4,4,4,4),
               dam = c(NA,NA,NA,NA,2,3,5,5,6,6),
               label = as.character(1:n_nr_animals))
mata_ex8p1_verify <- getA(ped = ped)
mata_ex8p1_verify

## 10 x 10 sparse Matrix of class "dsCMatrix"
##    [[ suppressing 10 column names '1', '2', '3' ... ]]
##
## 1  1.00 .      .      .    0.500 0.500 0.2500 0.2500 0.2500 0.2500
## 2  .    1.00 .      .    0.500 .    0.2500 0.2500 .      .
## 3  .    .    1.00 .      .    0.500 .    .    0.2500 0.2500
## 4  .    .    .    1.0 .      .    0.5000 0.5000 0.5000 0.5000
## 5  0.50 0.50 .      .    1.000 0.250 0.5000 0.5000 0.1250 0.1250
## 6  0.50 .    0.50 .    0.250 1.000 0.1250 0.1250 0.5000 0.5000
## 7  0.25 0.25 .    0.5 0.500 0.125 1.0000 0.5000 0.3125 0.3125
## 8  0.25 0.25 .    0.5 0.500 0.125 0.5000 1.0000 0.3125 0.3125
## 9  0.25 .    0.25 0.5 0.125 0.500 0.3125 0.3125 1.0000 0.5000
## 10 0.25 .    0.25 0.5 0.125 0.500 0.3125 0.3125 0.5000 1.0000
```

In the above result all elements which are 0 are represented by a dot.

## Problem 2: BLUP Animal Model

Use the following dataset to predict breeding values for all animals.

Table 3: Data for Animal Model

Animal	Sire	Dam	Herd	Observation
5	1	2	1	16.77
6	1	3	1	20.04
7	4	5	1	18.39
8	4	5	2	5.43
9	4	6	2	11.92
10	4	6	2	7.36

### Assumptions

- Random residuals are un-correlated and they all have equal variance  $\sigma_e^2$  which is assumed to be 24.
- The additive genetic variance  $\sigma_a^2$  is assumed to be 8.
- The pedigree is the same as in Problem 1. You can use `solve()` in R or `pedigreeemm::getAInv()` to invert  $A$ .

## Your Tasks

- Specify all components including expected values and variances of the animal model using the information from the dataset.
- Set up mixed model equations
- Solve mixed model equations for estimates of fixed effects and for predicted breeding values

## Solution

The animal model in general has the following form

$$y = X\beta + Za + e$$

where

- $y$  vector of length  $n$  of observations
- $\beta$  vector of length  $p$  of unknown fixed effects
- $X$   $n \times p$  design matrix linking fixed effects to observations
- $a$  vector of length  $q$  of unknown random breeding values
- $Z$   $n \times q$  design matrix linking breeding values to observations
- $e$  vector of length  $n$  of unknown random residuals

The expected values of the fixed effects  $\beta$  are the fixed effects themselves, hence  $E(\beta) = \beta$ . The expected values of the random components are defined as

$$\begin{aligned} E(a) &= 0 \\ E(e) &= 0 \\ E(y) &= X\beta \end{aligned}$$

The variances of fixed effects are always 0. Based on the assumption of uncorrelated residuals, we know that  $\text{var}(e) = I\sigma_e^2$ . Because, we have an animal model, we also know that  $\text{var}(a) = A\sigma_a^2$  where  $A$  corresponds to the numerator relationship matrix. In summary, the variances of the random effects are

$$\begin{aligned} \text{var}(a) &= G = A\sigma_a^2 \\ \text{var}(e) &= R = I\sigma_e^2 \\ \text{var}(y) &= ZGZ^T + R \end{aligned}$$

Inserting the information from the dataset into the model gives the following results.

- Vector of observations

$$y = \begin{bmatrix} 16.77 \\ 20.04 \\ 18.39 \\ 5.43 \\ 11.92 \\ 7.36 \end{bmatrix}$$

- Herds as fixed effects. We have two herds, hence vector  $\beta$  has length  $p = 2$ . Component  $\beta_1$  will denote the effect of the first herd and component  $\beta_2$  the effect of the second herd.

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

- Breeding values as random effects. In total, there are 10 animals in the pedigree and hence the length of the vector  $a$  is  $q = 10$ .

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix}$$

- The vector of random residuals is just

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}$$

The solutions for  $\hat{\beta}$  and  $\hat{a}$  are obtained by solving the mixed model equations. The mixed model equations for the animal model and under the assumptions specified above are defined as

$$\begin{bmatrix} X^T X & X^T Z \\ Z^T X & Z^T Z + \lambda * A^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} X^T y \\ Z^T y \end{bmatrix}$$

The single components are computed as

$$X^T X = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, X^T Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$Z^T Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2.0 & 0.5 & 0.5 & 0.0 & -1.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 1.5 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 1.5 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 3.0 & 1.0 & 1.0 & -1.0 & -1.0 & -1.0 & -1.0 \\ -1.0 & -1.0 & 0.0 & 1.0 & 3.0 & 0.0 & -1.0 & -1.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & -1.0 & 1.0 & 0.0 & 3.0 & 0.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & -1.0 & -1.0 & 0.0 & 2.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.0 & -1.0 & 0.0 & 0.0 & 2.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 55.20 \\ 24.71 \end{bmatrix}, Z^T y = \begin{bmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 16.77 \\ 20.04 \\ 18.39 \\ 5.43 \\ 11.92 \\ 7.36 \end{bmatrix}$$

Putting everything together into the mixed model equations leads to the following results

$$\begin{bmatrix} 3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 \\ 0.0 & 0.0 & 6.0 & 1.5 & 1.5 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.5 & 4.5 & 0.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.5 & 0.0 & 4.5 & 0.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 9.0 & 3.0 & 3.0 & -3.0 & -3.0 & -3.0 & -3.0 \\ 1.0 & 0.0 & -3.0 & -3.0 & 0.0 & 3.0 & 10.0 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & -3.0 & 0.0 & -3.0 & 3.0 & 0.0 & 10.0 & 0.0 & 0.0 & -3.0 & -3.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & -3.0 & -3.0 & 0.0 & 7.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 & 7.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & 0.0 & -3.0 & 0.0 & 0.0 & 7.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 7.0 \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} 55.20 \\ 24.71 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 16.77 \\ 20.04 \\ 18.39 \\ 5.43 \\ 11.92 \\ 7.36 \end{bmatrix}$$

The solutions are computed as

$$\begin{bmatrix} \hat{\beta} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} 3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 \\ 0.0 & 0.0 & 6.0 & 1.5 & 1.5 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.5 & 4.5 & 0.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.5 & 0.0 & 4.5 & 0.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 9.0 & 3.0 & 3.0 & -3.0 & -3.0 & -3.0 & -3.0 \\ 1.0 & 0.0 & -3.0 & -3.0 & 0.0 & 3.0 & 10.0 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & -3.0 & 0.0 & -3.0 & 3.0 & 0.0 & 10.0 & 0.0 & 0.0 & -3.0 & -3.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & -3.0 & -3.0 & 0.0 & 7.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 & 7.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & 0.0 & -3.0 & 0.0 & 0.0 & 7.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 7.0 \end{bmatrix}^{-1} \begin{bmatrix} 55.20 \\ 24.71 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 16.77 \\ 20.04 \\ 18.39 \\ 5.43 \\ 11.92 \\ 7.36 \end{bmatrix}$$

The solutions are

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \hat{a}_6 \\ \hat{a}_7 \\ \hat{a}_8 \\ \hat{a}_9 \\ \hat{a}_{10} \end{bmatrix} = \begin{bmatrix} 18.4889 \\ 8.1439 \\ -0.0116 \\ -0.3593 \\ 0.3478 \\ 0.0231 \\ -0.5448 \\ 0.5159 \\ -0.2377 \\ -0.6113 \\ 0.7704 \\ 0.1190 \end{bmatrix}$$