Livestock Breeding and Genomics - Solution 8

Peter von Rohr

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Problem 1: Compute Inbreeding Coefficients

Given the following pedigree.

Animal	Sire	Dam
1	NA	NA
2	NA	NA
3	1	NA
4	3	2
5	4	2
6	4	5

Your Task

Compute the inbreeding coefficients F_i for all animals using the matrix R that comes from the cholesky decomposition of the numerator relationship matrix A

Solution

The cholesky decomposition of A corresponds to

$$A = R \cdot R^T$$

where R is a lower triangular matrix. The diagonal elements $(A)_{ii}$ of A can be computed as the sum of the squared elements of all elements of Matrix R on row i.

$$(A)_{ii} = \sum_{j=1}^{i} (R)_{ij}^2$$

Therefore to get $(A)_{ii}$, we have to compute all elements of R on row i. Due to the relation of the cholesky-decomposition to the `LDL`-decomposition, we can write the matrix R as a product of the two matrices L and S

$$R = L \cdot S \tag{1}$$

where L is the lower triangular matrix from the `LDL`-decomposition and S is a diagonal matrix with diagonal elements $(S)_{ii}$ corresponding to

$$(S)_{ii} = \sqrt{(D)_{ii}}$$

where $(D)_{ii}$ correspond to the diagonal elements of the matrix D from the `LDL`-decomposition. From the `LDL`, we know that

$$(D)_{ii} = \frac{1}{2} - \frac{1}{4}(F_s + F_d) = 1 - \frac{1}{4}(A_{ss} + A_{dd})$$

Based on the decomposition of R into the product of L and S given in (1) and based on the property of the matrix L which is to be shown in the additional problem 3 of this exercise, we can derive the following rules for computing the elements of the matrix R

- Diagnoal elements $(R)_{ii}$:

$$(R)_{ii} = (S)_{ii} = \sqrt{(D)_{ii}} = \sqrt{1 - \frac{1}{4}(A_{ss} + A_{dd})}$$

where s and d are parents of animal i and $(A)_{ss}$ and $(A)_{dd}$ are diagonal elements of the numerator relationship matrix A.

- Off-diagonal elements $(R)_{ij}$ $(i \neq j)$:

$$(R)_{ij} = \frac{1}{2}(R_{sj} + R_{dj})$$

where s and d are parents of animal i.

The solution of this exercise is to compute $(A)_{ii}$ for all animals in the pedigree using the above described rules.

- $(A)_{11}$

$$(A)_{11} = (R)^2_{11} = 1$$

- $(A)_{22}$

$$(A)_{22} = (R)^2_{21} + (R)^2_{22} = 0 + 1 = 1$$

- $(A)_{33}$

$$(A)_{33} = (R)^2_{31} + (R)^2_{32} + (R)^2_{33} = 0.25 + 0 + 0.75 = 1$$

- $(A)_{44}$

$$\begin{aligned}(A)_{44} &= (R)^2_{41} + (R)^2_{42} + (R)^2_{43} + (R)^2_{44} \\ &= 0.0625 + 0.25 + 0.1875 + 0.5 = 1\end{aligned}$$

- $(A)_{55}$

$$\begin{aligned}(A)_{55} &= (R)^2_{51} + (R)^2_{52} + (R)^2_{53} + (R)^2_{54} + (R)^2_{55} \\ &= 0.015625 + 0.5625 + 0.046875 + 0.125 + 0.5 = 1.25\end{aligned}$$

- $(A)_{66}$

$$\begin{aligned}(A)_{66} &= (R)^2_{61} + (R)^2_{62} + (R)^2_{63} + (R)^2_{64} + (R)^2_{65} + (R)^2_{66} \\ &= 0.03515625 + 0.390625 + 0.1054688 + 0.28125 + 0.125 + 0.4375 = 1.375\end{aligned}$$

As a check, we can compute the inbreeding coefficients using the function `pedigreemm::inbreeding()`

```r
pedigreemm::inbreeding(ped = ped_sol10p01)
```

```
## [1] 0.000 0.000 0.000 0.000 0.250 0.375
```

Problem 2: Direct Construction of A^{-1}

Use the pedigree from problem 1 and the computed inbreeding coefficients from problem 1 to set up the inverse numerator relationship matrix A^{-1} using the general form of Henderson's rules for a pedigree with inbred animals. Compare your result using function `pedigreemm::getAInv()`.

Solution

As a pre-requisite, we assume that the pedigree is sorted such that parents come before progeny. Henderson's rules contain the following steps

- Start with a matrix A^{-1} where all elements are set to 0.
- Let d^i be the i-th diagonal element of D^{-1} for animal i, assuming i has parents s and d.
- Then add the following contributions to A^{-1}
    - d^i to the element (i,i)
    - $-d^i/2$ to the elements (s,i), (i,s), (d,i), (i,d)
    - $d^i/4$ to the elements (s,s), (s,d), (d,s), (d,d)

Applying these rules to the pedigree given in problem 1 leads to the following sequence of computations.

- Initialize the matrix A^{-1} with all 0

$$A^{-1} = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Animal 1

$$A^{-1} = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Animal 2

$$A^{-1} = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Animal 3

$$A^{-1} = \begin{bmatrix} 1.3333 & 0.0000 & -0.6667 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.6667 & 0.0000 & 1.3333 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

- Animal 4

$$A^{-1} = \begin{bmatrix} 1.3333 & 0.0000 & -0.6667 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.5000 & 0.5000 & -1.0000 & 0.0000 & 0.0000 \\ -0.6667 & 0.5000 & 1.8333 & -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -1.0000 & -1.0000 & 2.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

- Animal 5

$$A^{-1} = \begin{bmatrix} 1.3333 & 0.0000 & -0.6667 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 2.0000 & 0.5000 & -0.5000 & -1.0000 & 0.0000 \\ -0.6667 & 0.5000 & 1.8333 & -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.5000 & -1.0000 & 2.5000 & -1.0000 & 0.0000 \\ 0.0000 & -1.0000 & 0.0000 & -1.0000 & 2.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

- Animal 6

$$A^{-1} = \begin{bmatrix} 1.3333 & 0.0000 & -0.6667 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 2.0000 & 0.5000 & -0.5000 & -1.0000 & 0.0000 \\ -0.6667 & 0.5000 & 1.8333 & -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.5000 & -1.0000 & 3.0714 & -0.4286 & -1.1429 \\ 0.0000 & -1.0000 & 0.0000 & -0.4286 & 2.5714 & -1.1429 \\ 0.0000 & 0.0000 & 0.0000 & -1.1429 & -1.1429 & 2.2857 \end{bmatrix}$$

- Check with function `pedigreemm::getAinv()`

```r
pedigreemm::getAInv(ped = ped_sol10p01)
```

```
## 6 x 6 Matrix of class "dgeMatrix"
##            1    2          3          4          5         6
## 1  1.3333333  0.0 -0.6666667  0.0000000  0.0000000  0.000000
## 2  0.0000000  2.0  0.5000000 -0.5000000 -1.0000000  0.000000
## 3 -0.6666667  0.5  1.8333333 -1.0000000  0.0000000  0.000000
## 4  0.0000000 -0.5 -1.0000000  3.0714286 -0.4285714 -1.142857
## 5  0.0000000 -1.0  0.0000000 -0.4285714  2.5714286 -1.142857
## 6  0.0000000  0.0  0.0000000 -1.1428571 -1.1428571  2.285714
```

The difference between the computed matrix and the matrix from `pedigreemm::getAinv()`

$$\begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$