Livestock Breeding and Genomics - Solution 11

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Problem 1 Multivariate BLUP Animal Model

The table below contains data for pre-weaning gain (WWG) and post-weaning gain (PWG) for 5 beef calves.

Animal	Sex	Sire	Dam	WWG	PWG
4	Male	1	NA	4.5	6.8
5	Female	3	2	2.9	5.0
6	Female	1	2	3.9	6.8
7	Male	4	5	3.5	6.0
8	Male	3	6	5.0	7.5

The genetic variance-covariance matrix G_0 between the traits is

$$G_0 = \left[\begin{array}{cc} 20 & 18 \\ 18 & 40 \end{array}\right]$$

The residual variance-covariance matrix R_0 between the traits is

$$R_0 = \left[\begin{array}{cc} 40 & 11 \\ 11 & 30 \end{array}\right]$$

Your Task

Set up the mixed model equations for a multivariate BLUP analysis and compute the estimates for the fixed effects and the predictions for the breeding values.

Solution

The matrices X_1 and X_2 relate records of PWG and WWG to sex effects. For both traits, we have an effect for the male and female sex. Hence the vector β of fixed effects corresponds to

$$\beta = \left[\begin{array}{c} \beta_{M,WWG} \\ \beta_{F,WWG} \\ \beta_{M,PWG} \\ \beta_{F,PWG} \end{array}\right]$$

The matrices X_1 and X_2 are the same and correspond to

$$X_1 = X_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{array}\right]$$

Combining them to the multivariate version leads to

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Using the matrix X together with matrix $R = I_n \otimes R_0$ to get

$$X^TR^{-1}X = \begin{bmatrix} 0.083 & 0.000 & -0.031 & 0.000 \\ 0.000 & 0.056 & 0.000 & -0.020 \\ -0.031 & 0.000 & 0.111 & 0.000 \\ 0.000 & -0.020 & 0.000 & 0.074 \end{bmatrix}$$

Similarly to the fixed effects, we can put together the vector of breeding values a.

$$u = \begin{bmatrix} u_{1,WWG} \\ u_{2,WWG} \\ u_{3,WWG} \\ u_{4,WWG} \\ u_{5,WWG} \\ u_{6,WWG} \\ u_{7,WWG} \\ u_{8,WWG} \\ u_{1,PWG} \\ u_{2,PWG} \\ u_{3,PWG} \\ u_{4,PWG} \\ u_{5,PWG} \\ u_{6,PWG} \\ u_{7,PWG} \\ u_{8,PWG} \end{bmatrix}$$

The design matrices Z_1 and Z_2 are equal and they link observations to breeding values.

$$Z_1 = Z_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}$$

$$
Z = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Together with the numerator relationship matrix A we can get $G = G_0 \otimes A$ and from this $G^{-1} = G_0^{-1} \otimes A^{-1}$

$$
A^{-1} = \begin{bmatrix}
1.833 & 0.500 & 0.000 & -0.667 & 0.000 & -1.000 & 0.000 & 0.000 \\
0.500 & 2.000 & 0.500 & 0.000 & -1.000 & -1.000 & 0.000 & 0.000 \\
0.000 & 0.500 & 2.000 & 0.000 & -1.000 & 0.500 & 0.000 & -1.000 \\
-0.667 & 0.000 & 0.000 & 1.833 & 0.500 & 0.000 & -1.000 & 0.000 \\
0.000 & -1.000 & -1.000 & 0.500 & 2.500 & 0.000 & -1.000 & 0.000 \\
-1.000 & -1.000 & 0.500 & 0.000 & 0.000 & 2.500 & 0.000 & -1.000 \\
0.000 & 0.000 & 0.000 & -1.000 & -1.000 & 0.000 & 2.000 & 0.000 \\
0.000 & 0.000 & -1.000 & 0.000 & 0.000 & -1.000 & 0.000 & 2.000
\end{bmatrix}
$$

$$
G^{-1} = \begin{bmatrix}
0.15 & 0.04 & 0.00 & -0.06 & 0.00 & -0.08 & 0.00 & 0.00 & -0.07 & -0.02 & 0.00 & 0.03 & 0.00 & 0.04 & 0.00 & 0.00 \\
0.04 & 0.17 & 0.04 & 0.00 & -0.08 & -0.08 & 0.00 & 0.00 & -0.02 & -0.08 & -0.02 & 0.00 & 0.04 & 0.04 & 0.00 & 0.00 \\
0.00 & 0.04 & 0.17 & 0.00 & -0.08 & 0.04 & 0.00 & -0.08 & 0.00 & -0.02 & -0.08 & 0.00 & 0.04 & -0.02 & 0.00 & 0.04 \\
-0.06 & 0.00 & 0.00 & 0.15 & 0.04 & 0.00 & -0.08 & 0.00 & 0.03 & 0.00 & 0.00 & -0.07 & -0.02 & 0.00 & 0.04 & 0.00 \\
0.00 & -0.08 & -0.08 & 0.04 & 0.21 & 0.00 & -0.08 & 0.00 & 0.00 & 0.04 & 0.04 & -0.02 & -0.09 & 0.00 & 0.04 & 0.00 \\
-0.08 & -0.08 & 0.04 & 0.00 & 0.00 & 0.21 & 0.00 & -0.08 & 0.04 & 0.04 & -0.02 & 0.00 & 0.00 & -0.09 & 0.00 & 0.04 \\
0.00 & 0.00 & 0.00 & -0.08 & -0.08 & 0.00 & 0.17 & 0.00 & 0.00 & 0.00 & 0.00 & 0.04 & 0.04 & 0.00 & -0.08 & 0.00 \\
0.00 & 0.00 & -0.08 & 0.00 & 0.00 & -0.08 & 0.00 & 0.17 & 0.00 & 0.00 & 0.04 & 0.00 & 0.00 & 0.04 & 0.00 & -0.08 \\
-0.07 & -0.02 & 0.00 & 0.03 & 0.00 & 0.04 & 0.00 & 0.00 & 0.08 & 0.02 & 0.00 & -0.03 & 0.00 & -0.04 & 0.00 & 0.00 \\
-0.02 & -0.08 & -0.02 & 0.00 & 0.04 & 0.04 & 0.00 & 0.00 & 0.02 & 0.08 & 0.02 & 0.00 & -0.04 & -0.04 & 0.00 & 0.00 \\
0.00 & -0.02 & -0.08 & 0.00 & 0.04 & -0.02 & 0.00 & 0.04 & 0.00 & 0.02 & 0.08 & 0.00 & -0.04 & 0.02 & 0.00 & -0.04 \\
0.03 & 0.00 & 0.00 & -0.07 & -0.02 & 0.00 & 0.04 & 0.00 & -0.03 & 0.00 & 0.00 & 0.08 & 0.02 & 0.00 & -0.04 & 0.00 \\
0.00 & 0.04 & 0.04 & -0.02 & -0.09 & 0.00 & 0.04 & 0.00 & 0.00 & -0.04 & -0.04 & 0.02 & 0.11 & 0.00 & -0.04 & 0.00 \\
0.04 & 0.04 & -0.02 & 0.00 & 0.00 & -0.09 & 0.00 & 0.04 & -0.04 & -0.04 & 0.02 & 0.00 & 0.00 & 0.11 & 0.00 & -0.04 \\
0.00 & 0.00 & 0.00 & 0.04 & 0.04 & 0.00 & -0.08 & 0.00 & 0.00 & 0.00 & 0.00 & -0.04 & -0.04 & 0.00 & 0.08 & 0.00 \\
0.00 & 0.00 & 0.04 & 0.00 & 0.00 & 0.04 & 0.00 & -0.08 & 0.00 & 0.00 & -0.04 & 0.00 & 0.00 & -0.04 & 0.00 & 0.08
\end{bmatrix}
$$

Using the matrics X, Z, R^{-1} and G^{-1}, we can compute $Z^T R^{-1} X$ and $Z^T R^{-1} Z + G^{-1}$. These matrices define the right-hand side of the mixed model equations. But they are too be to be shown here.

The vector y of observations contains all observations of both traits

$$
y = \begin{bmatrix}
4.50 \\
2.90 \\
3.90 \\
3.50 \\
5.00 \\
6.80 \\
5.00 \\
6.80 \\
6.00 \\
7.50
\end{bmatrix}
$$

The right-hand side is computed as

$$
\begin{bmatrix}
X^T R^{-1} y \\
Z^T R^{-1} y
\end{bmatrix}
$$

The solutions are

$$
\begin{bmatrix}
\widehat{\beta_{M,WWG}} \\
\widehat{\beta_{F,WWG}} \\
\widehat{\beta_{M,PWG}} \\
\widehat{\beta_{F,PWG}} \\
\widehat{u_{1,WWG}} \\
\widehat{u_{2,WWG}} \\
\widehat{u_{3,WWG}} \\
\widehat{u_{4,WWG}} \\
\widehat{u_{5,WWG}} \\
\widehat{u_{6,WWG}} \\
\widehat{u_{7,WWG}} \\
\widehat{u_{8,WWG}} \\
\widehat{u_{1,PWG}} \\
\widehat{u_{2,PWG}} \\
\widehat{u_{3,PWG}} \\
\widehat{u_{4,PWG}} \\
\widehat{u_{5,PWG}} \\
\widehat{u_{6,PWG}} \\
\widehat{u_{7,PWG}} \\
\widehat{u_{8,PWG}}
\end{bmatrix}
=
\begin{bmatrix}
4.3609 \\
3.3973 \\
6.7999 \\
5.8803 \\
0.1509 \\
-0.0154 \\
-0.0784 \\
-0.0102 \\
-0.2703 \\
0.2758 \\
-0.3161 \\
0.2438 \\
0.2796 \\
-0.0076 \\
-0.1703 \\
-0.0127 \\
-0.4778 \\
0.5172 \\
-0.4790 \\
0.3920
\end{bmatrix}
$$

Problem 2 Comparison of Reliabilites

Compare the predicted breeding values and the reliabilites obtained as results of Problem 1 with results from two univariate analyses for the same traits are used in Problem 1. All parameters can be taken from Problem 1.

Solution

For a predicted breeding value $\hat{u}_i$, the reliability B_i is computed as

$$B_i = r^2_{u,\hat{u}} = 1 - \frac{PEV(\hat{u}_i)}{var(u_i)} = 1 - \frac{C^{22}_{ii}}{var(u_i)}$$

where C^{22}_{ii} are obtained from the inverse coefficient matrix of the mixed model equations. Just as a reminder, we can write the mixed model equations (MME) as

$$M \cdot s = r$$

with the vectors r and s corresponding to the right-hand side and to the unknowns of the MME. Hence

$$r = \begin{bmatrix} X^TR^{-1}y \\ Z^TR^{-1}y \end{bmatrix}$$

and

$$s = \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix}$$

The matrix C^{22} is taken from the inverse coefficient matrix.

$$M^{-1} = \left[\begin{array}{cc} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & Z^T R^{-1} Z + G^{-1} \end{array} \right]^{-1} = \left[\begin{array}{cc} C^{11} & C^{12} \\ C^{21} & C^{22} \end{array} \right]$$

For the two univariate analyses, we get the solutions for the fixed effects and the breeding values and their reliabilities as follows

- WWG: estimates s_{WWG} and reliabilites B_{WWG}

$$s_{WWG} = \left[\begin{array}{r} 4.3585 \\ 3.4044 \\ 0.0984 \\ -0.0188 \\ -0.0411 \\ -0.0087 \\ -0.1857 \\ 0.1769 \\ -0.2495 \\ 0.1826 \end{array} \right]$$

$$B_{WWG} = \left[\begin{array}{r} 0.0578 \\ 0.0158 \\ 0.0871 \\ 0.1446 \\ 0.1438 \\ 0.1154 \\ 0.1163 \\ 0.1553 \end{array} \right]$$

- PWG

$$s_{PWG} = \left[\begin{array}{r} 6.7979 \\ 5.8785 \\ 0.2769 \\ -0.0051 \\ -0.1707 \\ -0.0131 \\ -0.4709 \\ 0.5138 \\ -0.4644 \\ 0.3837 \end{array} \right]$$

$$B_{PWG} = \left[\begin{array}{r} 0.1022 \\ 0.0307 \\ 0.1547 \\ 0.2563 \\ 0.2529 \\ 0.2119 \\ 0.2154 \\ 0.2705 \end{array} \right]$$

The reliabilities from the bivariate analysis are obtained as

$$B = \begin{bmatrix} 0.0698 \\ 0.0202 \\ 0.1054 \\ 0.1748 \\ 0.1729 \\ 0.1424 \\ 0.1442 \\ 0.1858 \\ 0.1025 \\ 0.0308 \\ 0.1550 \\ 0.2569 \\ 0.2534 \\ 0.2124 \\ 0.2159 \\ 0.2710 \end{bmatrix}$$