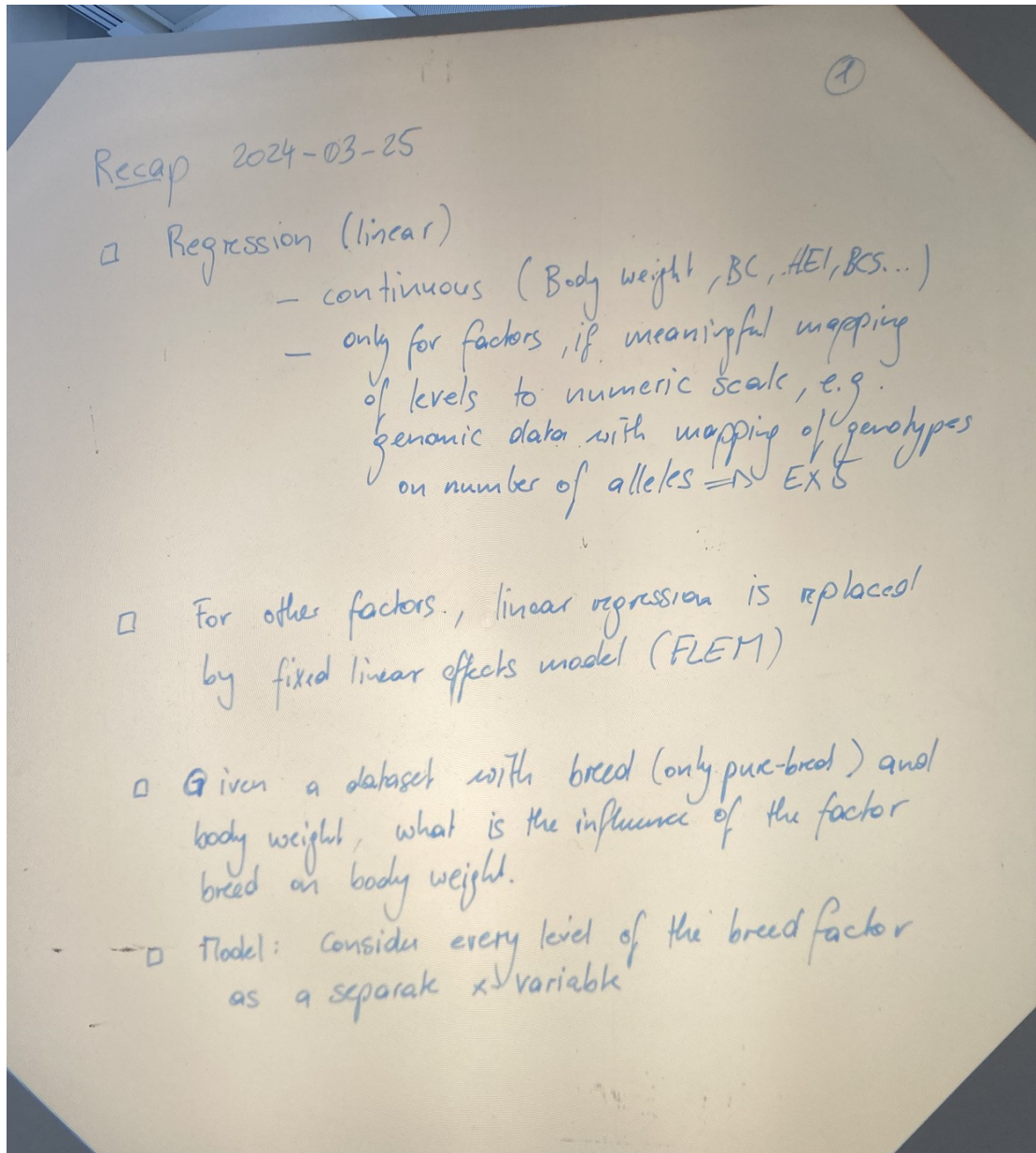


OHP Picture 1



OHP Picture 2

(2)

□ Model : Instead of just one regression-factor for breed with $E(y_i) = b_0 + b_1 x_i$ we have separate x-variables for every breed-level

$$\Rightarrow E(y_{ij}) = b_0 + b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 0$$

$$= b_0 + b_1 \cdot x_{1j} + b_2 \cdot x_{2j} + b_3 \cdot x_{3j}$$

expected value of body weight for animal j of breed i

Expected effect of breed Angus on body weight

Expected effect of breed Limousin on body weight

expected effect of Simmental on body weight

□ Dataset : Animal 1,7,6 with breed Angus, Lim, Si

Animal 1 $E(y_{11}) = b_0 + b_1 \cdot 1 + b_2 \cdot 0 + b_3 \cdot 0$

Animal 7 $E(y_{27}) = b_0 + b_1 \cdot 0 + b_2 \cdot 1 + b_3 \cdot 0$

Animal 6 $E(y_{36}) = b_0 + b_1 \cdot 0 + b_2 \cdot 0 + b_3 \cdot 1$

OHP Picture 3

③

Matrix-Vector Notation

- Data set sorted according to factor breed
- Define vector y as the vector of all body weights

$$y = \begin{bmatrix} 471 \\ 465 \\ 470 \\ \vdots \\ 491 \end{bmatrix}; \text{ vector } b = \begin{bmatrix} b_0 \\ \text{bangus} \\ \text{bimousin} \\ \text{bimunkel} \end{bmatrix}, e = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

Matrix $X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 1 \end{bmatrix}$ } Design-matrix
Links observations
to their fixed
breed effects.

- Model $y = X \cdot b + e$
- Goal: Estimate of unknown b using least squares
- SSQR = $(y - Xb)^T (y - Xb)$
- In regression: $\hat{b}_{OLS} = (X^T X)^{-1} X^T y$
- $(X^T X)$ cannot be inverted

OHP Picture 4

④

□ Structure $X^T X$

$$X^T X = \begin{bmatrix} N & \# \text{Angus} & \# \text{Lincoln} & \# \text{Simmental} \\ \# \text{An} & \# \text{An} & \emptyset & \emptyset \\ \# \text{Li} & \emptyset & \# \text{Li} & \emptyset \\ \# \text{Si} & \emptyset & \emptyset & \# \text{Si} \end{bmatrix}$$

□ $X^T X$ has linear dependence between columns, and therefore cannot be inverted. This can also be seen from Determinant of $X^T X$

In R: $\det(X^T X)$

□ Least Squares: Minimise $SSQR = (y - Xb)^T (y - Xb)$

Result of minimization are Least Squares Normal Equations

$$(X^T X) b^{(0)} = X^T y ; \text{ where } X^T X \text{ cannot be inverted}$$

□ Instead of using the non-existent inverse of $(X^T X)$, we can use a "generalized" inverse $(X^T X)^-$ of $(X^T X)$. Then a solution $b^{(0)}$ can be computed as

$$b^{(0)} = (X^T X)^- X^T y$$

⑤

□ Definition of a generalized inverse

• Given any matrix A , a generalized inverse (G) is the matrix that satisfies

$$A \cdot G \cdot A = A \Leftrightarrow (A \cdot G \cdot A)^T = A^T$$

Recall: the inverse A^{-1} was defined as

$$A^{-1}A = I \quad ; \quad A \cdot A^{-1} = I$$

$$A \cdot A^{-1} \cdot A = A$$

□ Why is generalize inverse (G) useful for us?

□ Given a system of equations, $A \cdot b = r$

the vector $b = G \cdot r$ is a solution, if G is a generalized inverse of A . So G is defined such that $AG \cdot A = A$, the solutions are given by

$b = G \cdot r$; pre-multiply with A

$$A \cdot b = A \cdot G \cdot r$$

$$\underline{A \cdot b} = \underline{A \cdot G \cdot A} \cdot r \Rightarrow A = AGA \text{ only true if } G \text{ is generalized inverse of } A$$

□ Our system of equations of interest are the least squares normal equations: (6)

$$\underbrace{(X^T X)}_A \mathbf{b} = \underbrace{X^T \mathbf{y}}_r$$

$$A \cdot \mathbf{b} = r$$

□ For a generalized inverse $(X^T X)^-$ of $X^T X$;

$\mathbf{b}^0 = (X^T X)^- X^T \mathbf{y}$ is a solution

□ (1) Generalized inverse $(X^T X)^-$ is not unique

□ (2) For any given $(X^T X)^-$; there can be an infinite number of solutions for the least squares normal equation

$$\mathbf{b} = (X^T X)^- X^T \mathbf{y} + [(X^T X)^- X^T X - I] \mathbf{z}$$

for any vector \mathbf{z}

$$(X^T X) \mathbf{b} = (X^T X) \cdot (X^T X)^- X^T \mathbf{y} + \underbrace{[(X^T X) (X^T X)^- (X^T X) - (X^T X)]}_{\begin{matrix} [X^T X & -X^T X] \end{matrix}} \cdot \mathbf{z}$$