OHP Picture 1

1

Recap: 2024-04-08

- □ Fixed linear effect model (FLEM) integrated discrete factors, such as breed into model
- □ Normal equations from least squares

$$(X^TX)\, b^{(0)} = X^T y$$

where X is a design matrix, linking levels of fixed effects to observations

and $b^{(0)}$ corresponds to one solution, computed as: $b^{(0)} = \underbrace{(X^TX)^-}_{G} X^T y = G X^T y$

- □ Non-uniqueness of $b^{(0)}$ is shown by

$\tilde{b} = b^{(0)} + \left(G(X^TX) - I\right) z$; for any vector z

is also a solution to normal equations

- □ Solution: Estimable functions of $b^{(0)}$

OHP Picture 2

(2)

- Example with small dataset on influence of breed on observation
- Model: $y = X \cdot b + e$

  y: vector of observations

  b: vector of breed levels: $b = \begin{bmatrix} b_0 \\ b_{Angus} \\ b_{Limousin} \\ b_{Simmental} \end{bmatrix} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$

  e: vector of random residuals

  X: design matrix $\underbrace{\text{linking breed levels to observations}}_{\text{"which observation comes from animal of a given breed"}}$

- Information from dataset to model:

$$y = \begin{bmatrix} 16 \\ 10 \\ \vdots \\ 13 \end{bmatrix}; \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_6 \end{bmatrix}; \quad X = \begin{array}{c} \\ 1 \\ 2 \\ \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} b_0 & b_A & b_L & b_S \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ & & & \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- $b^{(0)}$ computed as: $b^{(0)} = \begin{bmatrix} 13.5 \\ 1.5 \\ 13.5 \\ -1.5 \end{bmatrix}$; $\tilde{b} = b^{(0)} + (G(X^TX) - I) \cdot z$
- Check: $(X^TX)b^{(0)} = X^Ty \iff (X^TX)b^{(0)} - X^Ty = 0$

OHP Picture 3

(3)

□ Solution to normal equations:

$$\tilde{b}^{(1)} = \begin{bmatrix}14\\1\\13\\-2\end{bmatrix}\begin{matrix}\to \mu\\ \to \alpha_1\\ \to \alpha_2\\ \to \alpha_3\end{matrix} \; ; \quad \tilde{b}^{(4)} = \begin{bmatrix}1519.5\\-1504.5\\-1492.5\\-1507.5\end{bmatrix}\begin{matrix}\to \text{Intercept}\\ \to \text{effect of An}\\ \to \text{effect of Li}\\ \to \text{effect of Si}\end{matrix}$$

⇒ Components are differen

⇒ But some elements (functions of components) stay the same

	$\tilde{b}^{(1)}$	$\tilde{b}^{(4)}$
Example $\alpha_1 - \alpha_2$:	$1-13=-12$	$-1504.5+1492.5 = -12$

□ Write $\alpha_1 - \alpha_2$ as linear function (weighted sum) of solution $b^{(0)} = \begin{bmatrix}\mu\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}$:

$$\underbrace{\underbrace{\begin{bmatrix}0 & 1 & -1 & 0\end{bmatrix}}_{q^T}\overbrace{\begin{bmatrix}\mu\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}}^{b^{(0)}}}_{\alpha_1 - \alpha_2}$$

$q^T \cdot b^{(0)}$ is an estimable function of $b^{(0)}$

OHP Picture 4

4

- Example 2: $\mu + \alpha_1$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = q^T b^{(0)}$$

Ex 3: $\mu + 1/2 (\alpha_2 + \alpha_3)$

$$\begin{bmatrix} 1 & 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = q^T b^{(0)}$$

- Our interest: How are results of lm()-output computed?
- Estimate for intercept in lm-Output, corresponds to group-mean of Angus animals, because "Angus" corresponds to first level name in alphabet

OHP Picture 5

3

□ Breed Limousin:

$$b^{(0)} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \begin{array}{l} \\ \to \text{"Angus"} \\ \to \text{"Limousin"} \\ \to \text{"Simmental"} \end{array}$$

$\downarrow$

estimable function:

$$\begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = q^T b^{(0)}$$

□ Breed Simmental: $\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix} = q^T$

4