Estimable Functions

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Estimable Functions

Animal	Breed	Observation
1	Angus	16
2	Angus	10
3	Angus	19
4	Limousin	27
5	Simmental	11
6	Simmental	13

Model

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$$

$$\mathbf{y} = \begin{bmatrix} 16 \\ 10 \\ 19 \\ 27 \\ 11 \\ 13 \end{bmatrix}, \; \mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Normal Equations

$$X^T X b^{(0)} = X^T y$$

$$\begin{bmatrix} 6 & 3 & 1 & 2 \\ 3 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \mu^0 \\ \alpha_1^0 \\ \alpha_2^0 \\ \alpha_3^0 \end{bmatrix} = \begin{bmatrix} 96 \\ 45 \\ 27 \\ 24 \end{bmatrix}$$

A solution

$$b^{(0)} = \begin{bmatrix} 13.5 \\ 1.5 \\ 13.5 \\ -1.5 \end{bmatrix}$$

Solutions to Normal Equations

Elements of Solution	b_1^0	b_2^0	b_3^0	b_4^0
μ^0	14	15.5	15.25	1519.5
α_1^0	1	-0.5	-0.25	-1504.5
α_2^0	13	11.5	11.75	-1492.5
α_3^0	-2	-3.5	-3.25	-1507.5

Functions of Solutions

Linear Function	b_1^0	b_2^0	b_3^0	b_4^0
$\alpha_1^0 - \alpha_2^0$	-12.0	-12.0	-12.0	-12.0
$\mu^0 + \alpha_1^0$	15.0	15.0	15.0	15.0
$\mu^0 + 1/2(\alpha_2^0 + \alpha_3^0)$	19.5	19.5	19.5	19.5

- $\alpha_1^0 - \alpha_2^0$: estimate of the difference between breed effects for Angus and Simmental
- $\mu^0 + \alpha_1^0$: estimate of the general mean plus the breed effect of Angus
- $\mu^0 + 1/2(\alpha_2^0 + \alpha_3^0)$: estimate of the general mean plus mean effect of breeds Simmental and Limousin

Definition of Estimable Functions

$$\mathbf{q}^T\mathbf{b} = \mathbf{t}^T E(\mathbf{y})$$

- Why is $\mathbf{q}^T\mathbf{b}$ estimable? $\rightarrow$ invariance to solution $b^{(0)}$

$$\mathbf{q}^T\mathbf{b}^{(0)} = \mathbf{t}^T E(\mathbf{y}) = \mathbf{t}^T X\mathbf{b}^{(0)} = \mathbf{t}^T XGX^T y$$

where $\mathbf{XGX}^T$ is the same for all choices of $\mathbf{G}$

Examples

$$E(y_{1j} - y_{2j}) = \alpha_1 - \alpha_2$$

with $\mathbf{t}^T = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}$ and $\mathbf{q}^T = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$

$$E(y_{1j}) = \mu + \alpha_1$$

$$E(y_{2j}) = \mu + \alpha_2$$

$$E(y_{3j}) = \mu + \alpha_3$$

Property

Based on the definition, the following property can be derived

$$\mathbf{q}^t = \mathbf{t}^T \mathbf{X}$$

with the definition of an estimable function $\mathbf{q}^T \mathbf{b}$, we get

$$\mathbf{q}^T \mathbf{b} = \mathbf{t}^T E(\mathbf{y})$$

$$\mathbf{q}^T \mathbf{G} \mathbf{X}^T \mathbf{y} = \mathbf{t}^T \mathbf{X} \mathbf{G} \mathbf{X}^T \mathbf{y}$$

hence for any $\mathbf{G}$, $\mathbf{q}^t = \mathbf{t}^T \mathbf{X}$ which is helpful to find $\mathbf{q}$ for a given $\mathbf{t}$

Test

When we want to test whether a certain vector $\mathbf{q}$ can establish an estimable function, we can test wheter

$$\mathbf{q}^T\mathbf{H} = \mathbf{q}^T$$

with $\mathbf{H} = \mathbf{G}\mathbf{X}^T\mathbf{X}$

Setting $\mathbf{q}^T = \mathbf{t}^T\mathbf{X}$, we get

$$\mathbf{q}^T\mathbf{H} = \mathbf{t}^T\mathbf{X}\mathbf{H} = \mathbf{t}^T\mathbf{X} = \mathbf{q}^T$$