Linear Regression (Part 2)

Peter von Rohr

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Obtain Parameter Estimates in R

- Computations are tedious
- Use R builtin functions
- Assuming data is available in dataframe `tbl_reg` with columns `Body Weight` and `Breast Circumference`

```
lm_bw_bc <- lm(`Body Weight` ~ `Breast Circumference`,
               data = tbl_reg)
summary(lm_bw_bc)
```

The General Case

- Not only one x-variable, but many of them
- Parameter estimates can be derived the same way, but very cumbersome
- Use matrix-vector notation, for an example with two x-variables
- Define

$$\mathbf{X} = \begin{bmatrix} x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \\ . & . & . \\ . & . & . \\ . & . & . \\ x_{N0} & x_{N1} & x_{N2} \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_N \end{bmatrix}, \ \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ . \\ e_N \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

Linear Regression Model

$$\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}, \text{ with } E(\mathbf{y}) = \mathbf{X}\mathbf{b}$$

- General case with k x-variables

$$\mathbf{X} = \begin{bmatrix} x_{10} & x_{11} & . & x_{1k} \\ x_{20} & x_{21} & . & x_{2k} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ x_{N0} & x_{N1} & . & x_{Nk} \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ . \\ . \\ b_k \end{bmatrix}$$

Random Error Terms

- Properties of random error terms in vector $\mathbf{e}$

$$E(\mathbf{e}) = \mathbf{0}$$

$$var(\mathbf{e}) = E\left[\mathbf{e} - E(\mathbf{e})\right]\left[\mathbf{e} - E(\mathbf{e})\right]^T = E(\mathbf{e}\mathbf{e}^T) = \sigma^2 \mathbf{I}_N$$

Least Squares Estimates

$$\begin{aligned}\mathbf{e}^T\mathbf{e} &= [\mathbf{y} - E(\mathbf{y})]^T [\mathbf{y} - E(\mathbf{y})] \\ &= [\mathbf{y} - \mathbf{X}\mathbf{b}]^T [\mathbf{y} - \mathbf{X}\mathbf{b}] \\ &= \mathbf{y}^T\mathbf{y} - 2\mathbf{b}^T\mathbf{X}^T\mathbf{y} + \mathbf{b}^T\mathbf{X}^T\mathbf{X}\mathbf{b}\end{aligned}$$

- Setting

$$\frac{\partial \mathbf{e}^T\mathbf{e}}{\partial \mathbf{b}} = \mathbf{0}$$

- yields least squares normal equations

$$\mathbf{X}^T\mathbf{X}\hat{\mathbf{b}} = \mathbf{X}^T\mathbf{y}$$

Solution for Least Squares Estimators

$$\hat{\mathbf{b}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$