

OHP Picture 1

Recap 2023-03-06

□ Regression :

- response / target variable $\rightarrow y$
- predictor variable $\rightarrow x$

Body Weight y

x Breast Circumference

$L = \sum_{i=1}^N e_i^2$; determine b_0 (intercept) and b_1 (slope) of regression line such that L is minimal

$\hat{b}_0 = \bar{y} - \hat{b}_1 \bar{x}$

$\hat{b}_1 = \frac{(xy) - N\bar{x}\bar{y}}{(x^2) - N\bar{x}^2}$

Generalize :

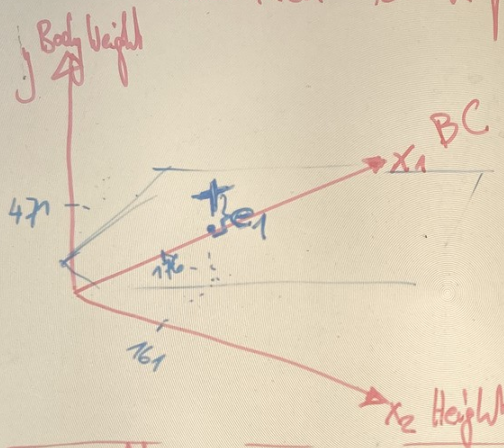
□ ~~to~~ to 2 predictor variables (Breast Circumference, ...)

OHP Picture 2

Generalize:

□ ~~to~~ to 2 predictor variables (Breast Circumference, Height.)

and then to k predictor variables



Expected value for Body weight of animal ^{height} i:
 $E(y_i) = b_0 + b_1 x_{i1} + b_2 x_{i2}$
 Unknown
 Breast Circum. of animal i

$$L = \sum_{i=1}^N [y_i - b_0 - b_1 x_{i1} - b_2 x_{i2}]^2 \rightarrow \min$$

- $b_0 =$
- $b_1 =$
- $b_2 =$

K - Predictor Variables

OHP Picture 3

K - Predictor Variables

Ani	Body Weight y	BC x ₁	HEI x ₂	x ₃	x ₄	... x _k
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$$L = \sum_{i=1}^N [y_i - b_0 - b_1 x_{i1} - b_2 x_{i2} - b_3 x_{i3} - \dots - b_k x_{ik}]^2 \rightarrow \text{min}$$

$b_0 =$
 $b_1 =$
 $b_2 =$
 $b_k =$

Simplified Notation : Matrix - Vector

• Matrix : $X = \begin{bmatrix} x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \\ \vdots & \vdots & \vdots \\ x_{i0} & x_{i1} & x_{i2} \\ \vdots & \vdots & \vdots \\ x_{n0} & x_{n1} & x_{n2} \end{bmatrix}$

$\underbrace{\quad}_{\text{all}=1 \text{ Intercept}}$ $\underbrace{\quad}_{\text{Breast circumference}}$ $\rightarrow \text{Height}$

Vectors : $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$; $\beta = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \end{bmatrix}$ \rightarrow Intercept Regression

Simplified Notation: Matrix-Vector

• Matrix: $X = \begin{bmatrix} X_{10} & X_{11} & X_{12} \\ X_{20} & X_{21} & X_{22} \\ \vdots & \vdots & \vdots \\ X_{i0} & X_{i1} & X_{i2} \\ \vdots & \vdots & \vdots \\ X_{n0} & X_{n1} & X_{n2} \end{bmatrix}$

$\underbrace{\quad}_{\text{all}=1}$ Intercept $\quad \underbrace{\quad}_{\text{Breadth}}$ circumference $\quad \rightarrow$ Height

Vectors: $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$; $b = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$

$\left. \begin{array}{l} \text{Intercept} \\ \text{Regression coefficient for BC} \\ \text{Regression coefficient for Height} \end{array} \right\}$ Unbran
 Body height

$e = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$

$E(y_1) = b_0 + b_1 x_{11} + b_2 x_{12}$
 $E(y_2) = b_0 + b_1 x_{21} + b_2 x_{22}$

$E(y_i) = b_0 + b_1 x_{i1} + b_2 x_{i2} \rightarrow E(y_i)$

$\rightarrow E(y) = Xb$
 $y = Xb + e \Rightarrow e = y - Xb$

OHP Picture 5

How estimate b :

□ Elements in vector b are unknown
 \Rightarrow estimate from data using Least Squares

$$L = e^T e = [e_1 \ e_2 \ e_3 \ \dots \ e_N] \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix} = e_1^2 + e_2^2 + \dots + e_N^2$$
$$L = \sum_{i=1}^N e_i^2 = e_1^2 + e_2^2 + \dots + e_N^2$$

$$\begin{aligned} L = e^T e &= (y - Xb)^T \cdot (y - Xb) \\ &= (y^T - (Xb)^T) \cdot (y - Xb) \\ &= (y^T - b^T X^T) \cdot (y - Xb) \\ &= \underbrace{y^T y}_{y^T y} - \underbrace{y^T X b}_{y^T X b} - \underbrace{b^T X^T y}_{(b^T X^T y)^T = y^T X b} + b^T X^T X b \\ &= y^T y - 2 y^T X b + b^T X^T X b \end{aligned}$$

Least Squares: Find vector b such that

OHP Picture 6

Least Squares: Find vector b such that

$$\left\{ \frac{\partial L}{\partial b_0}, \frac{\partial L}{\partial b_1}, \frac{\partial L}{\partial b_2}, \dots, \frac{\partial L}{\partial b_n} \right\} \text{ are all } 0$$

Compute gradient of L with respect to b

$$\frac{\partial L}{\partial b} = \begin{bmatrix} \frac{\partial L}{\partial b_0} \\ \frac{\partial L}{\partial b_1} \\ \vdots \\ \frac{\partial L}{\partial b_n} \end{bmatrix} \left\{ \begin{array}{l} L = y^T y - 2y^T X b + b^T X^T X b \\ \frac{\partial L}{\partial b} = 0 - 2y^T X + 2b^T X^T X \\ \text{Find } b \text{ such that } \frac{\partial L}{\partial b} = 0 \\ \Rightarrow -2y^T X + 2b^T X^T X = 0 \\ \hat{b}^T X^T X = y^T X \end{array} \right. \left. \begin{array}{l} \frac{\partial x^n}{\partial x} = nx^{n-1} \\ = 1 \end{array} \right.$$

$$(X^T X) \hat{b} = X^T y \Rightarrow \text{Normal Equations}$$

Given that $(X^T X)$ can be inverted:

$$\hat{b} = (X^T X)^{-1} X^T y$$

OHP Picture 7

Include also discrete variables as predictors

□ Example : - Breed as an influence on body weight.

- Breed is a discrete variable with fixed levels like
{ Angus, Simmental, Limousin }

□ Data Set

Animal	Body Weight	Breed
1	471	1
2	.	1
⋮		
10	541	2

□ Expected body weight for animal i :

$$E(y_i) = b_0 + b_1 \cdot x_i$$

intercept regression coefficient

breed code

Eg. Animal 1: $E(y_1) = b_0 + b_1 \cdot 1$
Animal 10: $E(y_{10}) = b_0 + b_1 \cdot 2$