## Inverse Numerator Relationship Matrix

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### Structure of $A^{-1}$

▶ Look at a simple example of A and  $A^{-1}$ 

Table 1: Pedigree Used To Compute Inverse Numerator Relationship Matrix

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2

## Numerator Relationship Matrix A

$$A = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.5000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.5000 & 0.5000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.5000 \\ 0.5000 & 0.5000 & 0.0000 & 1.0000 & 0.2500 \\ 0.0000 & 0.5000 & 0.5000 & 0.2500 & 1.0000 \end{bmatrix}$$

## Inverse Numerator Relationship Matrix $A^{-1}$

$$A^{-1} = \begin{bmatrix} 1.5000 & 0.5000 & 0.0000 & -1.0000 & 0.0000 \\ 0.5000 & 2.0000 & 0.5000 & -1.0000 & -1.0000 \\ 0.0000 & 0.5000 & 1.5000 & 0.0000 & -1.0000 \\ -1.0000 & -1.0000 & 0.0000 & 2.0000 & 0.0000 \\ 0.0000 & -1.0000 & -1.0000 & 0.0000 & 2.0000 \end{bmatrix}$$

### Conclusions

- $ightharpoonup A^{-1}$  has simpler structure than A itself
- ► Non-zero elements only at positions of parent-progeny and parent-mate positions
- Parent-mate positions are positive, parent-progeny are negative

### Henderson's Rules

▶ Based on LDL-decomposition of *A* 

$$A = L * D * L^T$$

- where L Lower triangular matrix D Diagonal matrix
- ► Why?
  - ▶ matrices L and D can be inverted directly, we 'll see how . . .
  - ightharpoonup construct  $A^{-1} = (L^T)^{-1} * D^{-1} * L^{-1}$

## Example

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 1.0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

 $\rightarrow$  Verify that  $A = L * D * L^T$ 

## Decomposition of True Breeding Value

ightharpoonup True breeding value  $(u_i)$  of animal i

$$u_i = \frac{1}{2}u_s + \frac{1}{2}u_d + m_i$$

▶ Do that for all animals in pedigree

## Decomposition for Example

$$u_1 = m_1$$

$$u_2 = m_2$$

$$u_3 = m_3$$

$$u_4 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_4$$

$$u_5 = \frac{1}{2}u_3 + \frac{1}{2}u_2 + m_5$$

### Matrix Vector Notation

- ▶ Define vectors *u* and *m* as
- $\triangleright$  Coefficients of  $u_s$  and  $u_d$  into matrix P

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}, m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix}, P = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 0.0 \end{bmatrix}$$

Result: Decomposition of true breeding values

$$u = P \cdot u + m$$

## Decomposition of Variance

 $\triangleright$  Analogous decomposition of  $var(u_i)$ 

#### based on the decomposition of u\_i

$$var(u_i) = var(1/2u_s + 1/2u_d + m_i)$$
  
=  $var(1/2u_s) + var(1/2u_d) + \frac{1}{2} * cov(u_s, u_d) + var(m_i)$   
=  $1/4var(u_s) + 1/4var(u_d) + \frac{1}{2} * cov(u_s, u_d) + var(m_i)$ 

From the definition of A

$$var(u_i) = (1 + F_i)\sigma_u^2$$
  
 $var(u_s) = (1 + F_s)\sigma_u^2$   
 $var(u_d) = (1 + F_d)\sigma_u^2$   
 $cov(u_s, u_d) = (A)_{sd}\sigma_u^2 = 2F_i\sigma_u^2$ 

## Variance of Mendelian Sampling Terms

- $\blacktriangleright$  What is  $var(m_i)$ ?
- ▶ Solve equation for  $var(u_i)$  for  $var(m_i)$

$$var(m_i) = var(u_i) - 1/4var(u_s) - 1/4var(u_d) - 2 * cov(u_s, u_d)$$

Insert definitions from A

Diagonal Element of Matrix D for animal i

$$var(m_i) = (1 + F_i)\sigma_u^2 - 1/4(1 + F_s)\sigma_u^2 - 1/4(1 + F_d)\sigma_u^2 - \frac{1}{2} * 2 * F_i\sigma_u^2$$
$$= \left(\frac{1}{2} - \frac{1}{4}(F_s + F_d)\right)\sigma_u^2$$

 $var(m_4) = (1/2 - 1/4(F_1 + F_2)) \times gma_u^2$ ; obtain F\_1 and F\_2 from matrix A

True, for both parents s and d of animal i are known

### **Unknown Parents**

Only parent s of animal i is known

$$u_i = \frac{1}{2}u_s + m_i$$

$$var(m_i) = \left(1 - \frac{1}{4}(1 + F_s)\right)\sigma_u^2$$

$$= \left(\frac{3}{4} - \frac{1}{4}F_s\right)\sigma_u^2$$

Both parents are unknown

$$u_i = m_i$$
 $var(m_i) = \sigma_u^2$ 

## Recursive Decomposition

So far: Simple Decomposition: u = P \* u + m

ightharpoonup True breeding values of s and d can be decomposed into

$$u_{s} = \frac{1}{2}u_{ss} + \frac{1}{2}u_{ds} + m_{s}$$
$$u_{d} = \frac{1}{2}u_{sd} + \frac{1}{2}u_{dd} + m_{d}$$

where ss sire of s
ds dam of s
sd sire of d
dd dam of d

## Example

▶ Add animal 6 with parents 4 and 5 to our example pedigree

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2
6	4	5

## First Step Of Decomposition

new animal 6:

$$u_{1} = m_{1}$$

$$u_{2} = m_{2}$$

$$u_{3} = m_{3}$$

$$u_{4} = \frac{1}{2}u_{1} + \frac{1}{2}u_{2} + m_{2}$$

$$u_{5} = \frac{1}{2}u_{3} + \frac{1}{2}u_{2} + m_{5}$$

$$u_{6} = \frac{1}{2}u_{4} + \frac{1}{2}u_{5} + m_{6}$$

## **Decompose Parents**

$$u_{1} = m_{1}$$

$$u_{2} = m_{2}$$

$$u_{3} = m_{3}$$

$$u_{4} = \frac{1}{2}m_{1} + \frac{1}{2}m_{2} + m_{4}$$

$$u_{5} = \frac{1}{2}m_{3} + \frac{1}{2}m_{2} + m_{5}$$

$$u_{6} = \frac{1}{2}\left(\frac{1}{2}(u_{1} + u_{2}) + m_{4}\right) + \frac{1}{2}\left(\frac{1}{2}(u_{3} + u_{2}) + m_{5}\right) + m_{6}$$

$$= \frac{1}{4}(u_{1} + u_{2}) + \frac{1}{2}m_{4} + \frac{1}{4}(u_{3} + u_{2}) + \frac{1}{2}m_{5} + m_{6}$$

## Decompose Grand Parents

Only animal 6 has true breeding values for grand parents

$$u_6 = \frac{1}{4}(u_1 + u_2) + \frac{1}{2}m_4 + \frac{1}{4}(u_3 + u_2) + \frac{1}{2}m_5 + m_6$$

$$= \frac{1}{4}m_1 + \frac{1}{4}m_2 + \frac{1}{4}m_3 + \frac{1}{4}m_2 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6$$

$$= \frac{1}{4}m_1 + \frac{1}{2}m_2 + \frac{1}{4}m_3 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6$$

## Summary

#### Full recursive decomposition:

==> all breeding values are decomposed into sums of mendelian sampling terms

$$u_{1} = m_{1}$$

$$u_{2} = m_{2}$$

$$u_{3} = m_{3}$$

$$u_{4} = \frac{1}{2}m_{1} + \frac{1}{2}m_{2} + m_{4}$$

$$u_{5} = \frac{1}{2}m_{3} + \frac{1}{2}m_{2} + m_{5}$$

$$u_{6} = \frac{1}{4}m_{1} + \frac{1}{2}m_{2} + \frac{1}{4}m_{3} + \frac{1}{2}m_{4} + \frac{1}{2}m_{5} + m_{6}$$

### Matrix-Vector Notation

#### Matrix L is lower-triangular

▶ Use vectors *u* and *m* again

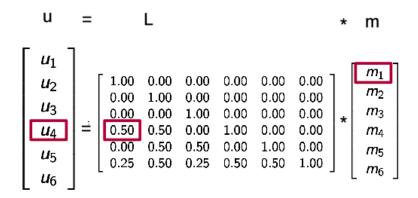
$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}, \ m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}, \ L = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 0.50 & 0.50 & 0.50 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.50 & 0.50 & 0.50 & 0.50 & 1.00 \end{bmatrix}$$

 $\triangleright$  Result of recursive decomposition of  $u_i$ 

$$u = L \cdot m$$

### Property of *L*

▶ Meaning of Element  $(L)_{ij}$  of Matrix L:



## Property of *L* II

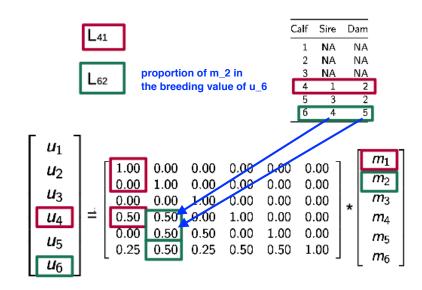
# lower triangle of L

- ▶ Element  $(L)_{ij}$  (i > j) is the proportion of  $m_j$  in  $u_i$
- ▶ Given: *i* has parents *s* and *d*
- $m{m}_j$  can only come from  $u_s$  and  $u_d$ , because  $u_i = 1/2u_s + 1/2u_d + m_i$
- ▶ The proportion of  $m_j$  in  $u_i$  is half the proportion of  $m_j$  in  $u_s$  and half the proportion of  $m_j$  in  $u_d$

$$\rightarrow L_{ij} = \frac{1}{2}L_{sj} + \frac{1}{2}L_{dj}$$

## Example

► L<sub>41</sub>, L<sub>62</sub>



## Variance From Recursive Decomposition

$$var(u) = var(L \cdot m)$$
  
=  $L \cdot var(m) \cdot L^{T}$ 

where var(m) is the variance-covariance matrix of all components in vector m.

- ▶ covariances of components  $m_i$ ,  $cov(m_i, m_i) = 0$  for  $i \neq j$
- $\triangleright$   $var(m_i)$  computed as shown before

### Result

• variance-covariance matrix var(m) can be written as  $D * \sigma_u^2$  where D is diagnoal

**Text** 

$$\rightarrow A = L \cdot D \cdot L^T$$

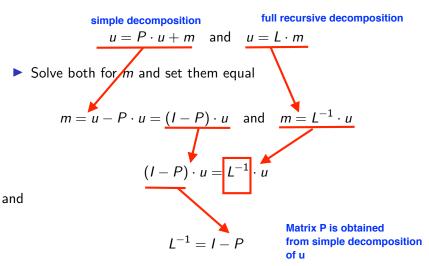
### Inverse of A Based on L and D

- ▶ Matrix A was decomposed into  $A = L \cdot D \cdot L^T$
- ightharpoonup Get  $A^{-1}$  as  $A^{-1} = (L^T)^{-1}D^{-1}L^{-1}$
- $ightharpoonup D^{-1}$  is diagonal again with elements

$$(D^{-1})_{ii} = 1/(D)_{ii}$$

#### Inverse of L

Compute m based on the two decompositions of u



## Example

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2

### Matrix $D^{-1}$

► Because *D* is diagonal

(D)44 =  $(1/2 - 1/4(F_-1 + F_-2))$ where F\_1 is inbreeding coefficient of animal 1 and F\_2 is the inbreeding coefficient of animal 2 and they are obtained from the diagonal elements of matrix A, as F\_1 = (A)11 - 1 and F\_2 = (A)22 - 1

$$D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

ightharpoonup We get  $D^{-1}$  as

$$D^{-1} = \left[ \begin{array}{ccccc} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{array} \right]$$

### Matrix $L^{-1}$

▶ Use 
$$L^{-1} = I - P$$

► Matrix *P* from simple decomposition

$$P = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 0.0 \end{bmatrix}$$

► Therefore

$$L^{-1} = I - P = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.5 & -0.5 & 0.0 & 1.0 \end{bmatrix}$$

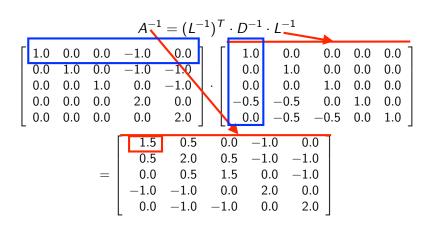
## Decomposition of $A^{-1}$ I

$$A^{-1} = (L^{-1})^T \cdot D^{-1} \cdot L^{-1}$$

$$(L^{-1})^T \cdot D^{-1}$$

$$\begin{bmatrix}
1.0 & 0.0 & 0.0 & -0.5 & 0.0 \\
0.0 & 1.0 & 0.0 & -0.5 & -0.5 \\
0.0 & 0.0 & 1.0 & 0.0 & -0.5 \\
0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 1.0 & 0.0
\end{bmatrix} \cdot \begin{bmatrix}
1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -1.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
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0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0$$

## Decomposition of $A^{-1}$ II



### Henderson's Rules

#### rule for animal 4

- Both Parents Known

  - ▶ add 2 to the diagonal-element (i, i)▶ add -1 to off-diagonal elements (s, i), (i, s), (d, i) and (i, d)
  - ▶ add  $\frac{1}{2}$  to elements (s, s), (d, d), (s, d), (d, s)
- Only One Parent Known
  - ▶ add  $\frac{4}{3}$  to diagonal-element (i, i)
  - ▶ add  $-\frac{2}{3}$  to off-diagonal elements (s, i), (i, s)
  - ightharpoonup add  $\frac{1}{2}$  to element (s,s)
- Both Parents Unknown
  - $\triangleright$  add 1 to diagonal-element (i, i)
- Valid without inbreeding

Rule for animals 1, 2 and 3

```
Apply Henderson's rules for our example pedigree:
Preparation step: Start with empty matrix A^{-1}, all elements are 0
Animal 1: both parents unknown ==> add to empty matrix +1 to diagonal element (1,1)
Animal 2: ...
Animal 3: ...
Animal 4: parents 1 and 2:
 add +2 to (4,4),
 subtract 1 from (1,4), (2,4), (4,1) and element (4,2)
 add 1/2 to (1.1). (2,2), (1,2) and (2,1)
Animal 5:
Animal 6:
Add up all the contributions:
(A)11: +1 + 1/2 = 1.5
```

In R: verify the computation of A inverse by the function getAlnv()