Livestock Breeding and Genomics - Solution 6

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Problem 1: Numerator Relationship Matrix

Construct the numerator relationship matrix A for the following pedigree and verify the result using the function `getA()` from package `pedigreemm`.

Table 1: Pedigree For Constructing Numerator Relationship Matrix

Animal	Sire	Dam
5	1	2
6	1	3
7	4	5
8	4	5
9	4	6
10	4	6

Solution

The numerator relationship is constructed using the follownig step-wise procedure. The following rules are used to compute the single elements.

- **Case 1**: If both parents s and d of animal i are known then
  - the diagonal element $(A)_{ii}$ corresponds to: $(A)_{ii} = 1 + F_i = 1 + \frac{1}{2}(A)_{sd}$ and
  - the offdiagonal element $(A)_{ji}$ is computed as: $(A)_{ji} = \frac{1}{2}((A)_{js} + (A)_{jd})$
  - because A is symmetric $(A)_{ji} = (A)_{ij}$
- **Case 2**: If only one parent s is known and assumed unrelated to the mate
  - $(A)_{ii} = 1$
  - $(A)_{ij} = (A)_{ji} = \frac{1}{2}((A)_{js}$
- **Case 3**: If both parents are unknown
  - $(A)_{ii} = 1$
  - $(A)_{ij} = (A)_{ji} = 0$

Step 1 First, we extend the pedigree given in Table 1. All animals without parents are added at the top of the pedigree. This results in the matrix shown in Table 2.

Table 2: Extended Pedigree

Animal	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA

4	NA	NA
5	1	2
6	1	3
7	4	5
8	4	5
9	4	6
10	4	6

Because the pedigree in Table 2 is already ordered such that parents are before offspring, we can directly go to the next step.

Step 2 We start with an empty numerator relationship matrix A. The matrix A has dimensions 10×10

$$A = \begin{bmatrix} & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \end{bmatrix}$$

Step 3 The single elements of A are computed according to the rules listed above.

The computation is started with animal 1. The first element is always the diagonal-element that corresponds to animal that we are currently looking at. For animal 1 the diagonal element is $(A)_{11}$. Because animal 1 has not parents, we are in case 3 for the diagonal element. If an animal has unknown parents, it also means that the animals's inbreeding coefficient F_i is 0. Hence

$$(A)_{11} = 1$$

Now we have the first element of our numerator relationship matrix.

$$A = \begin{bmatrix} 1.00 & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \end{bmatrix}$$

Step 4 The next elements that need to be computed are the off-diagnoal element on row 1. Elements A_{12}, A_{13} and A_{14} correspond to the additive genetic relationship between animal 1 and animals 2, 3 and 4. Because animals 2, 3 and 4 all have unknown parents, we are for all three elements in case 3, hence we can state

$$(A)_{12} = (A)_{13} = (A)_{14} = 0$$

For the remaining elements of the first row of A, the elements correspond to the additive genetic relationship between animal 1 and animals 5 to 10. Because animals 5 to 10 all have known parents, we have to use case 1 in the above formulated rules.

$$(A)_{15} = \frac{1}{2}\left((A)_{11} + (A)_{12}\right) = \frac{1}{2}(1+0) = 0.5$$

$$(A)_{16} = \frac{1}{2}\left((A)_{11} + (A)_{13}\right) = \frac{1}{2}(1+0) = 0.5$$

$$(A)_{17} = \frac{1}{2}\left((A)_{14} + (A)_{15}\right) = \frac{1}{2}(0+0.5) = 0.25$$

$$(A)_{18} = \frac{1}{2}\left((A)_{14} + (A)_{15}\right) = \frac{1}{2}(0+0.5) = 0.25$$

$$(A)_{19} = \frac{1}{2}\left((A)_{14} + (A)_{16}\right) = \frac{1}{2}(0+0.5) = 0.25$$

$$(A)_{110} = \frac{1}{2}\left((A)_{14} + (A)_{16}\right) = \frac{1}{2}(0+0.5) = 0.25$$

As a result, we have the first row of A

$$A = \left[\begin{array}{cccccccccc} 1.00 & 0.00 & 0.00 & 0.00 & 0.50 & 0.50 & 0.25 & 0.25 & 0.25 & 0.25 \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \end{array}\right]$$

Step 5 Copy the first row to the first column

$$A = \left[\begin{array}{cccccccccc} 1.00 & 0.00 & 0.00 & 0.00 & 0.50 & 0.50 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.00 & & & & & & & & & \\ 0.00 & & & & & & & & & \\ 0.00 & & & & & & & & & \\ 0.50 & & & & & & & & & \\ 0.50 & & & & & & & & & \\ 0.25 & & & & & & & & & \\ 0.25 & & & & & & & & & \\ 0.25 & & & & & & & & & \\ 0.25 & & & & & & & & & \end{array}\right]$$

Step 6 Continue the same way with rows 2 to 10

Verification We first have to specify the pedigree, before being able to get the numerator relationship matrix

```
n_nr_animals <- 10
suppressPackageStartupMessages( library(pedigreemm) )
ped <- pedigree(sire = c(NA,NA,NA,NA,1,1,4,4,4,4),
                dam = c(NA,NA,NA,NA,2,3,5,5,6,6),
                label = as.character(1:n_nr_animals))
mata_ex8p1_verify <- getA(ped = ped)
mata_ex8p1_verify
```

```
## 10 x 10 sparse Matrix of class "dsCMatrix"

##    [[ suppressing 10 column names '1', '2', '3' ... ]]

##
## 1  1.00 .    .    .   0.500 0.500 0.2500 0.2500 0.2500 0.2500
## 2  .    1.00 .    .   0.500 .     0.2500 0.2500 .      .
## 3  .    .    1.00 .   .     0.500 .      .      0.2500 0.2500
## 4  .    .    .    1.0 .     .     0.5000 0.5000 0.5000 0.5000
## 5  0.50 0.50 .    .   1.000 0.250 0.5000 0.5000 0.1250 0.1250
## 6  0.50 .    0.50 .   0.250 1.000 0.1250 0.1250 0.5000 0.5000
## 7  0.25 0.25 .    0.5 0.500 0.125 1.0000 0.5000 0.3125 0.3125
## 8  0.25 0.25 .    0.5 0.500 0.125 0.5000 1.0000 0.3125 0.3125
## 9  0.25 .    0.25 0.5 0.125 0.500 0.3125 0.3125 1.0000 0.5000
## 10 0.25 .    0.25 0.5 0.125 0.500 0.3125 0.3125 0.5000 1.0000
```

In the above result all elements which are 0 are represented by a dot.

Problem 2: BLUP Animal Model

Use the following dataset to predict breeding values for all animals.

Table 3: Data for Animal Model

Animal	Sire	Dam	Herd	Observation
5	1	2	1	16.77
6	1	3	1	20.04
7	4	5	1	18.39
8	4	5	2	5.43
9	4	6	2	11.92
10	4	6	2	7.36

Assumptions

- Random residuals are un-correlated and they all have equal variance σ_e^2 which is assumed to be 24.
- The additive genetic variance σ_a^2 is assumed to be 8.
- The pedigree is the same as in Problem 1. You can use `solve()` in R or `pedigreemm::getAInv()` to invert A.

Your Tasks

- Specify all components including expected values and variances of the animal model using the information from the dataset.
- Set up mixed model equations
- Solve mixed model equations for estimates of fixed effects and for predicted breeding values

Solution

The animal model in general has the following form

$$y = X\beta + Za + e$$

where

y	vector of length n of observations
β	vector of length p of unknown fixed effects
X	$n \times p$ design matrix linking fixed effects to observations
a	vector of length q of unkown random breeding values
Z	$n \times q$ design matrix linking breeding values to observations
e	vector of length n of unknown random residuals

The expected values of the fixed effects β are the fixed effects themselves, hence $E(\beta) = \beta$. The expected values of the random components are defined as

$$\begin{aligned} E(a) &= 0 \\ E(e) &= 0 \\ E(y) &= X\beta \end{aligned}$$

The variances of fixed effects are always 0. Based on the assumption of uncorrelated residuals, we know that $var(e) = I\sigma_e^2$. Because, we have an animal model, we also know that $var(a) = A\sigma_a^2$ where A corresponds to the numerator relationship matrix. In summary, the variances of the random effects are

$$\begin{aligned} var(a) &= G = A\sigma_a^2 \\ var(e) &= R = I\sigma_e^2 \\ var(y) &= ZGZ^T + R \end{aligned}$$

Inserting the information from the dataset into the model gives the following results.

- Vector of observations

$$y = \begin{bmatrix} 16.77 \\ 20.04 \\ 18.39 \\ 5.43 \\ 11.92 \\ 7.36 \end{bmatrix}$$

- Herds as fixed effects. We have two herds, hence vector β has length $p = 2$. Component β_1 will denote the effect of the first herd and component β_2 the effect of the second herd.

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

- Breeding values as random effects. In total, there are 10 animals in the pedigree and hence the length of the vector a is $q = 10$.

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix}$$

- The vector of random residuals is just

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}$$

The solutions for $\hat{\beta}$ and $\hat{a}$ are obtained by solving the mixed model equations. The mixed model equations for the animal model and under the assumptions specified above are defined as

$$\begin{bmatrix} X^TX & X^TZ \\ Z^TX & Z^TZ + \lambda * A^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} X^Ty \\ Z^Ty \end{bmatrix}$$

The single components are computed as

$$X^TX = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, X^TZ = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$Z^TZ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2.0 & 0.5 & 0.5 & 0.0 & -1.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 1.5 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 1.5 & 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 3.0 & 1.0 & 1.0 & -1.0 & -1.0 & -1.0 & -1.0 \\ -1.0 & -1.0 & 0.0 & 1.0 & 3.0 & 0.0 & -1.0 & -1.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & -1.0 & 1.0 & 0.0 & 3.0 & 0.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & -1.0 & -1.0 & 0.0 & 2.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.0 & -1.0 & 0.0 & 0.0 & 2.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 55.20 \\ 24.71 \end{bmatrix}, \; Z^T y = \begin{bmatrix} 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 16.77 \\ 20.04 \\ 18.39 \\ 5.43 \\ 11.92 \\ 7.36 \end{bmatrix}$$

Putting everything together into the mixed model equations leads to the following results

$$\begin{bmatrix} 3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 \\ 0.0 & 0.0 & 6.0 & 1.5 & 1.5 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.5 & 4.5 & 0.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.5 & 0.0 & 4.5 & 0.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 9.0 & 3.0 & 3.0 & -3.0 & -3.0 & -3.0 & -3.0 \\ 1.0 & 0.0 & -3.0 & -3.0 & 0.0 & 3.0 & 10.0 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & -3.0 & 0.0 & -3.0 & 3.0 & 0.0 & 10.0 & 0.0 & 0.0 & -3.0 & -3.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & -3.0 & -3.0 & 0.0 & 7.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 & 7.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & 0.0 & -3.0 & 0.0 & 0.0 & 7.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 7.0 \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} 55.20 \\ 24.71 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 16.77 \\ 20.04 \\ 18.39 \\ 5.43 \\ 11.92 \\ 7.36 \end{bmatrix}$$

The solutions are computed as

$$\begin{bmatrix} \hat{\beta} \\ \hat{a} \end{bmatrix} = \begin{bmatrix} 3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 1.0 \\ 0.0 & 0.0 & 6.0 & 1.5 & 1.5 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.5 & 4.5 & 0.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.5 & 0.0 & 4.5 & 0.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 9.0 & 3.0 & 3.0 & -3.0 & -3.0 & -3.0 & -3.0 \\ 1.0 & 0.0 & -3.0 & -3.0 & 0.0 & 3.0 & 10.0 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 \\ 1.0 & 0.0 & -3.0 & 0.0 & -3.0 & 3.0 & 0.0 & 10.0 & 0.0 & 0.0 & -3.0 & -3.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & -3.0 & -3.0 & 0.0 & 7.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & -3.0 & 0.0 & 0.0 & 7.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & 0.0 & -3.0 & 0.0 & 0.0 & 7.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & -3.0 & 0.0 & -3.0 & 0.0 & 0.0 & 0.0 & 7.0 \end{bmatrix}^{-1} \begin{bmatrix} 55.20 \\ 24.71 \\ 0.00 \\ 0.00 \\ 0.00 \\ 0.00 \\ 16.77 \\ 20.04 \\ 18.39 \\ 5.43 \\ 11.92 \\ 7.36 \end{bmatrix}$$

The solutions are

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \\ \hat{a}_6 \\ \hat{a}_7 \\ \hat{a}_8 \\ \hat{a}_9 \\ \hat{a}_{10} \end{bmatrix} = \begin{bmatrix} 18.4889 \\ 8.1439 \\ -0.0116 \\ -0.3593 \\ 0.3478 \\ 0.0231 \\ -0.5448 \\ 0.5159 \\ -0.2377 \\ -0.6113 \\ 0.7704 \\ 0.1190 \end{bmatrix}$$