OHP Picture 1

Recap:

- Prediction of breeding values
  - Regression method: $\hat{u} = b \cdot (y - \mu)$
  - Different data scenarios:
    1) own performance
    2) repeated observation
    3) offspring: sires used daughters records

$y = \mu + u + e$

- Problems
  - not flexible, cannot use all available observation
  - → Method that can use all available information

- Prediction of breeding values follows the same uniform principle:
  1. Correction of phenotypes with some suitable population mean. The reason for this is that we have defined breeding values as deviations.
  2. Corrected information $(y-\mu)$ is multiplied by an appropriate factor called b.

OHP Picture 2

Selection Index:

- later used to predict aggregate genotype (Gesamtzuchtwert)
- Generic method useful to combine different sources of information into one summary quantity

- Before the method BLUP was invented, selection index theory was used to predict breeding values for a single trait.

- Idea:
  - Combine all sources of phenotypic information, e.g.
    - > own performance (one, repeated)
    - > half-sib
    - > full-sib
    - > offspring
    - > parent
  - Assume: general principle of prediction: $\hat{u} = b\overbrace{(y-\mu)}^{y^*}$

$$= by^*$$

$\Rightarrow$ Index of merit I:

$$I = b_1(y_1-\mu_1) + b_2(y_2-\mu_2) + \dots + b_k(y_k-\mu_k)$$

$$= b_1 y_1^* + b_2 y_2^* + \dots + b_k y_k^*$$

$$= b^T y^*$$ (vector-dot product with

$$b^T = \begin{bmatrix} b_1 & b_2 & b_3 & \dots & b_k \end{bmatrix}; \quad y^* = \begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_k^* \end{bmatrix}$$

OHP Picture 3

Prediction of breeding value u_i for animal i for a single trait:

$\hat{u}_i = I = b^T \cdot y^*$ with y^*: vector of all information corrected for appropriate population mean

b^T: vector of unknown index weights

- Goal: $\hat{u}_i$ should be such that it predicts u_i as good as possible $\Rightarrow$ Error $(u_i - \hat{u}_i)$ should be minimal
- In selection index theory the error is quantified by the prediction error variance: $var(u_i - \hat{u}_i)$

$\Rightarrow$ Minimize $R = var(u_i - \hat{u}_i)$

$$R = var(u_i - \hat{u}_i) = var(u_i - I) = var(u_i - b^T \cdot y^*)$$

$$= var(u_i) + var(b^T y^*) - 2\,cov(u_i, (y^*)^T \cdot b)$$

$$= \sigma_u^2 + b^T \cdot \underbrace{var(y^*)}_{} \cdot b - 2\, b^T cov(u_i, (y^*)^T)$$

$$var\left(\begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_k^* \end{bmatrix}\right) = \begin{bmatrix} var(y_1^*) & cov(y_1^*, y_2^*) & \dots \\ cov(y_2^*, y_1^*) & var(y_2^*) & \dots \\ & & \end{bmatrix} = P$$

OHP Picture 4

□ Var of a vector corresponds to a variance-covariance matrix.

Example: vector $y^* = \begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_k^* \end{bmatrix}$

$$P = var(y^*) = \begin{bmatrix} var(y_1^*) & cov(y_1^*, y_2^*) & cov(y_1^*, y_3^*) & \cdots & cov(y_1^*, y_k^*) \\ cov(y_2^*, y_1^*) & var(y_2^*) & cov(y_2^*, y_3^*) & \cdots & cov(y_2^*, y_k^*) \\ \vdots & & & & \\ cov(y_k^*, y_1^*) & cov(y_k^*, y_2^*) & \cdots & & var(y_k^*) \end{bmatrix}$$

- symmetric matrix
- it has k rows and k columns

□ $G = cov(u, (y^*)^T) = \begin{bmatrix} cov(u, y_1^*) \\ cov(u, y_2^*) \\ \vdots \\ cov(u, y_k^*) \end{bmatrix}$

□ $R = \sigma_u^2 + b^T P b + 2\, b^T \cdot G$

□ Minimization: $\underbrace{\frac{\partial R}{\partial b}}_{\text{Gradient}} = \emptyset = \begin{bmatrix} \frac{\partial R}{\partial b_1} \\ \frac{\partial R}{\partial b_2} \\ \vdots \end{bmatrix}$

OHP Picture 5

- Minimize R:

$$R = \sigma_u^2 + b^T P b - 2 b^T G$$

$$\frac{\partial R}{\partial b} = 0 + 2 b^T P - 2 G^T = 0$$

$$\Leftrightarrow \quad 2Pb - 2G = 0$$

$$Pb = G$$

; because P is a variance-covariance matrix, it is positive-definite, i.e. its inverse exist

$$\Rightarrow \quad \underbrace{P^{-1}P}_{I} b = P^{-1} \cdot G$$

$$b = P^{-1} \cdot G$$

- Summary: Selection Index theory provides a method to predict breeding values using all available information.

- Problem:
  - Find appropriate population means to correct phenotypic observations
  - How to find μ such that $y^* = (y - \mu)$ for all possible data scenarios

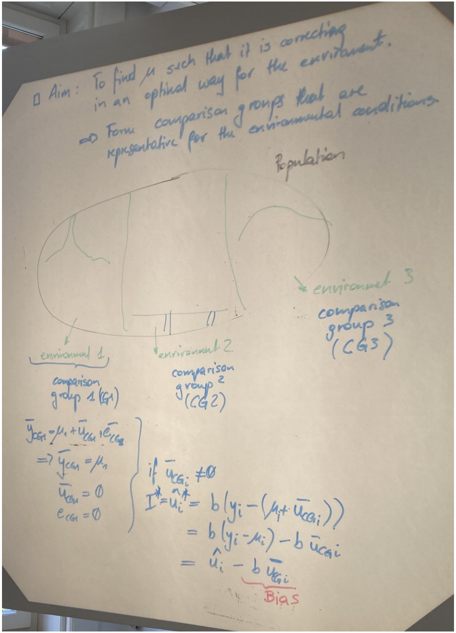
- Requirement for μ: Based on model: $y = \mu + u + e$

  compute average $\bar{y}$: $\bar{y} = \bar{\mu} + \bar{u} + \bar{e}$

  because u and e are deviations, $\bar{u} = \bar{e} = 0$

  $\Rightarrow \bar{y} = \mu$

OHP Picture 6



OHP Picture 7

□ Solution to Bias-Problem in selection index was found when BLUP method was invented:

B : Best → error (prediction error variance) minimal

L : Linear → Linear combination of data

U : unbiased → expected value $E[\hat{u}] = u$

P : Prediction → Breeding values are treated as random effects, in English/American Literature, the term "prediction" is always used for random effects, whereas for fixed effects, the term "estimation" is used ⇒ BLUE

"Vorhersage"

"Schätzung"

Prediction of breeding values (Zuchtwertschätzung)

□ BLUP uses Linear models

- Simple linear fixe effect model
- Example: In example data set: (Ziel größe)
  - > Weaning weight is the response variable (y)
  - > Herd, (Sex), Animal, Sire, dam as predictor variables (beschreibende Variable)
- $y_{ij} = \mu + herd_j + e_{ij}$

What is the effect of the herd on the response variable

OHP Picture 8

- In a fixed linear effect model, only fixed effects can be included. In example:

  $y_{ij} = \mu + herd_j + e_{ij}$

- Fit the model to data:

  for animal 12: $2.61 = \mu + 1\,herd_1 + e_{12,1}$

  13: $2.31 = \mu + herd_1 + e_{13,1}$

  ⋮

  27: $3.16 = \mu + 1\,herd_2 + e_{27,2}$

  with unknown intercept μ and unknown herd effects $herd_1$ for herd 1 and $herd_2$ for herd 2

- Result will be fixed numbers for μ, $herd_1$ and $herd_2$ such that the sum of the squared residuals (e_{ij}) is minimal. ⇒ Least Squares Type of estimation.

- It is not possible to include any variances of an effect into a fixed linear model

⟹ Solution are mixed linear effect models (LME)

OHP Picture 9

- Mixed linear effect models contain fixed and random effects
- Example:

$$\underbrace{y_{ijk}}_{\text{observation } k \text{ for animal } i \text{ in herd } j} = \underbrace{\mu}_{\text{intercept}} + \underbrace{\beta_j}_{\text{fixed effect of herd } j} + \underbrace{u_i}_{\text{random breeding value for animal } i} + \underbrace{e_{ijk}}_{\text{random residual}}$$

- Fit the model to the data:

Animal 12: $2.61 = \mu + \beta_1 + u_{12} + e_{12,1,1}$

13: $2.31 = \mu + \beta_1 + u_{13} + e_{13,1,1}$

⋮

27: $3.16 = \mu + \beta_2 + u_{27} + e_{27,2,1}$

Goal:
- Estimates for fixed effects β_1 and β_2
- Predictions for breeding values: $u_1, \dots, u_{27}$
- $\boxed{\text{Estimate of } \sigma_u^2 \text{ and } \sigma_e^2}$ → not possible with Least Squares

OHP Picture 10

□ Notation for system of equations

- Scalar notation

$$\underbrace{\begin{bmatrix} y_{12,1,1} \\ y_{13,1,1} \\ \vdots \\ y_{27,2,1} \end{bmatrix}}_{y} \begin{matrix} = \mu + \beta_1 + u_{12} + e_{12,1,1} \\ = \mu + \beta_1 + u_{13} + e_{13,1,1} \\ \\ = \mu + \beta_2 + u_{27} + e_{27,2,1} \end{matrix}$$

- Matrix vector notation:
  - > combine all observations: $y_{12,1,1}$ --- $y_{27,2,1}$ into a single vector $y = \begin{bmatrix} y_{12,1,1} \\ y_{13,1,1} \\ \vdots \\ y_{27,2,1} \end{bmatrix}$
  - > fixed effects in vector $\beta = \begin{bmatrix} \mu \\ \beta_1 \\ \beta_2 \end{bmatrix}$
  - > random breeding values $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{27} \end{bmatrix}$
  - > random error terms $e = \begin{bmatrix} e_{12,1,1} \\ \vdots \\ e_{27,2,1} \end{bmatrix}$

OHP Picture 11

- Final form:

$$y = X\beta + Zu + e$$

$$\begin{array}{c} y \\ \begin{bmatrix} y_{12,1,1} \\ y_{15,1,1} \\ \vdots \\ \vdots \\ y_{27,2,1} \end{bmatrix} \end{array} = \begin{array}{c} X \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \beta \\ \begin{bmatrix} \mu \\ \beta_1 \\ \beta_2 \end{bmatrix} \end{array} + \begin{array}{c} Z \\ \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0 \cdots 1\,0 \cdots 0 \\ 0 \cdots\cdots 0\,1\,0 \cdots 0 \\ 0\,0 \cdots\cdots \\ \vdots \\ 0 \cdots\cdots 1 \end{bmatrix} \end{array} \begin{array}{c} u \\ \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{27} \end{bmatrix} \end{array} + e$$

From Scalar notation:

$$2.61 = y_{12,1,1} = \mu + \beta_1 + \boxed{1 \cdot u_{12}} + e_{12,1,1}$$

$$= \underbrace{1 \cdot \mu + 1 \cdot \beta_1 + 0 \cdot \beta_2}_{} + u_{12} + e_{12,1,1}$$

encoded into the dot product of the first row of matrix X times the vector β.

$$3.16 = y_{27,2,1} = 1 \cdot \mu + 0 \cdot \beta_1 + 1 \beta_2 + u_{27} + e_{27,2,1}$$

OHP Picture 12

- Linear Mixed Effect model all variances and covariances of random effects and all expected values must be specified.

- Model : $y = X\beta + Zu + e$

y → random, $X\beta$ → fixed, Zu → random, e → random

$E[\beta] = \beta$

$var[\beta] = \emptyset$

Breeding values u are defined as deviations

$\Rightarrow E[u] = \emptyset \rightarrow \begin{bmatrix} \emptyset \\ \vdots \\ \emptyset \end{bmatrix}$

vector $u - \begin{bmatrix} u_1 \\ \vdots \\ u_{27} \end{bmatrix}$

Residuals are also deviations

$\Rightarrow E[e] = \emptyset$

$$\Rightarrow E[y] = E[X\beta + Zu + e] = E[X\beta] + E[Zu] + E[e]$$
$$= X E[\beta] + Z E[u] + E[e]$$
$$= X\beta + Z\emptyset + \emptyset = X\beta$$

OHP Picture 13

□ Variances:

$var(u) = G$

↓ variance-covariance matrix of random locality values

$var(e) = R$

↓ variance-covariance matrix of random errors

$\text{-}\ cov(u, e^T) = 0 \;;\; cov(\beta, u^T) = 0 \;,\; cov(\beta, e^T) = 0$

$$\Rightarrow var(y) = var(X\beta + Zu + e)$$

$$= var(X\beta) + var(Zu) + var(e)$$

$$= X\underbrace{var(\beta)}_{0}X^T + Z\,var(u)\,Z^T + var(e)$$

$$= ZGZ^T + R = V$$