# OHP Picture 1

Recap:

- Problem of correcting systematic environment with
  - regression
  - selection index

⇒ Solution: Use BLUP together with a mixed linear effect model to estimate systematic environment (herd, sex, season,) effects and to predict breeding values as random effects simultaneously from the same data set.

⇒ Model: $\underbrace{y = X\beta + Zu + e}_{\text{1. part}} \rightarrow \underbrace{\begin{cases} E(u) = & var(u) = .. \\ E(e) = & var(e) = .. \\ E(y) = & var(y) = .. \end{cases}}_{\text{2. part}}$

with $\beta$, $u$ and $e$ being unkown

Fit data, obtain $\hat{\beta}$ as estimates for fixed effects and $\hat{u}$ as predictions of breeding values

- Problem with $\hat{u}$ and $\hat{\beta}$: depend on $V^{-1}$ where $V = var(y)$ {Practical evaluations, $y$ can have the length of $10^{7}$
- Solution: Mixed model equations

# OHP Picture 2

Mixed Model Equations (MME)

$$\begin{bmatrix} X^TR^{-1}X & X^TR^{-1}Z \\ Z^TR^{-1}X & Z^TR^{-1}Z+G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^TR^{-1}y \\ Z^TR^{-1}y \end{bmatrix}$$

- Remember: $y = X\beta + Zu + e$ ; $var(e) = R$ ; variance-covariance matrix of residuals

known

in MME, we need $R^{-1}$

1. We assume that residual terms $e_1, \ldots, e_i, \ldots, e_N$, they have the same variance $\Rightarrow var(e_i) = \sigma_e^2$

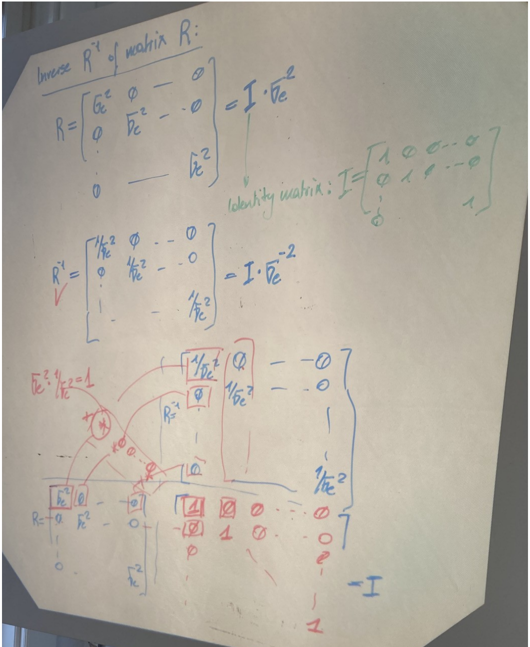
   $var(e_1) = var(e_2) = \ldots = var(e_i) = \ldots = var(e_N) = \sigma_e^2$

   vector $e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix}$ where $N$ is the number of observations in data set.

2. Covariance between two residual effects is $\emptyset$: $cov(e_i, e_j) = \emptyset$ for $i \neq j$

$$R = \begin{bmatrix} var(e_1) & cov(e_1,e_2) & \cdots \\ cov(e_2,e_1) & var(e_2) & \cdots \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} \sigma_e^2 & \emptyset & \emptyset & \cdots & \emptyset \\ \emptyset & \sigma_e^2 & \emptyset & \cdots & \emptyset \\ \vdots & & & & \vdots \\ \emptyset & \cdots & & & \sigma_e^2 \end{bmatrix}$$

# OHP Picture 3



# OHP Picture 4

Matrix G:

- Model definition $var(u) = G = \begin{bmatrix} var(u_1) & cov(u_1,u_2) & \cdots \\ cov(u_2,u_1) & var(u_2) & \cdots \\ & & \end{bmatrix}$

- Depends on relationship between breeding values

- First example:
  - Sire model
  - Breeding values of sires are random effects $s$
  - ~~Cow~~ Female animals and (male) animals without offspring do not get breeding values.

- Model: $y = X\beta + Zs + e$; $E[e] = 0$

  $E[s] = 0$

  $E[y] = X\beta$

  $var[e] = R = I \cdot \sigma_e^2$

  $var[s] = G$

  $var[y] = ZGZ^T + R$

- Data: $y = \begin{bmatrix} 2.61 \\ 2.31 \\ \vdots \\ 3.16 \end{bmatrix}$ $s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$

  $\beta = \begin{bmatrix} herd\ 1 \\ herd\ 2 \end{bmatrix}$ $e = \begin{bmatrix} e_1 \\ \vdots \\ e_{16} \end{bmatrix}$

# OHP Picture 5

- Design matrices for sire model:

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

- Solutions using Mixed Model Equations

$$\begin{bmatrix} X^TR^{-1}X & X^TR^{-1}Z \\ Z^TR^{-1}X & Z^TR^{-1}Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{s} \end{bmatrix} = \begin{bmatrix} X^TR^{-1}y \\ Z^TR^{-1}y \end{bmatrix}$$

- Simplify general MME, assuming $R^{-1} = I \cdot \sigma_e^{-2}$

$$\begin{bmatrix} \boxed{X^T \cdot I \cdot \sigma_e^{-2} \cdot X} & X^T \cdot I \cdot \sigma_e^{-2} Z \\ Z^T I \sigma_e^{-2} X & Z^T I \sigma_e^{-2} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{s} \end{bmatrix} = \begin{bmatrix} X^T I \sigma_e^{-2} y \\ Z^T I \sigma_e^{-2} y \end{bmatrix}$$

$$X^T \cdot I * \underbrace{\sigma_e^{-2}}_{\text{scalar}} \cdot X = \underbrace{X^T \cdot I}_{X^T} \cdot X * \sigma_e^{-2} = X^T X * \sigma_e^{-2}$$

# OHP Picture 6

- MME:

$$\begin{bmatrix} X^TX*\sigma_e^{-2} & X^TZ*\sigma_e^{-2} \\ Z^TX*\sigma_e^{-2} & Z^TZ*\sigma_e^{-2}+G^{-1}\end{bmatrix}\begin{bmatrix}\hat{\beta}\\ \hat{s}\end{bmatrix} = \begin{bmatrix} X^Ty*\sigma_e^{-2} \\ Z^Ty*\sigma_e^{-2}\end{bmatrix} \quad // *\sigma_e^{2}$$

$$\begin{bmatrix} X^TX & X^TZ \\ Z^TX & Z^TZ+G^{-1}*\sigma_e^{2}\end{bmatrix}\begin{bmatrix}\hat{\beta}\\ \hat{s}\end{bmatrix} = \begin{bmatrix} X^Ty \\ Z^Ty\end{bmatrix}$$

$$G^{-1}*\sigma_e^{2} \rightarrow I*\sigma_s^{-2} \rightarrow I*\lambda \text{ with } \lambda = \frac{\sigma_e^2}{\sigma_s^2}$$

- $G = var(s) = \begin{bmatrix} var(s_1) & cov(s_1,s_2) & cov(s_1,s_3) \\ cov(s_2,s_1) & var(s_2) & cov(s_2,s_3) \\ cov(s_3,s_1) & cov(s_3,s_2) & var(s_3)\end{bmatrix}$

- From data set (pedigree) we can see that sires 1-3 do not have any known parents
  $\Rightarrow$ cov between their effects is $\emptyset$: $cov(s_1,s_2) = cov(s_1,s_3) = cov(s_2,s_3) = 0$

  $var(s_i) = \sigma_s^2$

  $\Rightarrow G = I*\sigma_s^2 \Rightarrow G^{-1} = I*\sigma_s^{-2}$

# OHP Picture 7

- MME:

$$\begin{bmatrix} X^TX * \sigma_e^{-2} & X^TZ * \sigma_e^{-2} \\ Z^TX * \sigma_e^{-2} & Z^TZ * \sigma_e^{-2} + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{s} \end{bmatrix} = \begin{bmatrix} X^Ty * \sigma_e^{-2} \\ Z^Ty * \sigma_e^{-2} \end{bmatrix} \quad // * \sigma_e^{2}$$

$$\begin{bmatrix} X^TX & X^TZ \\ Z^TX & Z^TZ + G^{-1} * \sigma_e^{2} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{s} \end{bmatrix} = \begin{bmatrix} X^Ty \\ Z^Ty \end{bmatrix}$$

$I * \sigma_s^{-2} \rightarrow I * \lambda$ with $\lambda = \frac{\sigma_e^2}{\sigma_s^2}$

- $G = var(s) = \begin{bmatrix} var(s_1) & cov(s_1, s_2) & cov(s_1, s_3) \\ cov(s_2, s_1) & var(s_2) & cov(s_2, s_3) \\ cov(s_3, s_1) & cov(s_3, s_2) & var(s_3) \end{bmatrix}$

- From data set (pedigree) we can see that sires 1-3 do not have any known parents

$\Rightarrow$ cov between their effects is $\emptyset$: $cov(s_1, s_2)$
$= cov(s_1, s_3)$
$= cov(s_2, s_3) = \emptyset$

$var(s_i) = \sigma_s^2$

$\Rightarrow G = I * \sigma_s^2 \Rightarrow G^{-1} = I * \sigma_s^{-2}$

# OHP Picture 8