OHP Picture 1

Recap:

- Example of a linear mixed effects model (lme) to predict breeding values
  - ⇒ Sire model, only sires get breeding values
  - ⇒ sire effects are considered as random effects
- Model: $y = X\beta + Zs + e$

$$E = \begin{bmatrix} e \\ s \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ X\beta \end{bmatrix} \Leftrightarrow \begin{cases} E[e] = 0 \\ E[s] = 0 \\ E[y] = X\beta \end{cases}$$

$$var \begin{bmatrix} y \\ s \\ e \end{bmatrix} = \begin{bmatrix} ZGZ^T + R & ZG & R \\ GZ^T & G & 0 \\ R & 0 & R \end{bmatrix}; \quad var(e) = R = I \cdot \sigma_e^2, \quad var(s) = G = I \cdot \sigma_s^2$$

- Solution for $\hat{\beta}$ and $\hat{s}$ were obtained by solving the MME:

$$\underbrace{\begin{bmatrix} X^TX & X^TZ \\ Z^TX & Z^TZ + I \cdot \lambda \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \hat{\beta} \\ \hat{s} \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} X^Ty \\ Z^Ty \end{bmatrix}}_{r}$$

$$b = M^{-1} r$$

OHP Picture 2

From Sire Model to Animal Model

- Sire model provides predicted breeding values only for sires.
- Sire model only accounts for relationships between sires
- New: · Animal model predicts breeding values for all animals in the pedigree
  · Takes into account all relationships between animals
- BLUP. Animal Model:

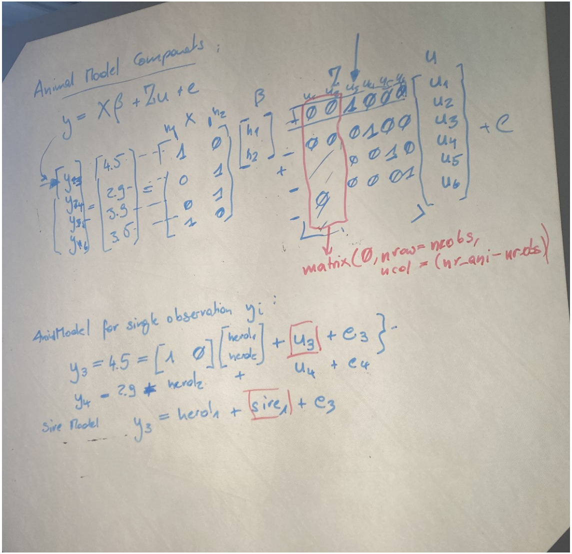
$$y = X\beta + Zu + e$$

u : vector of random breeding values for all animals in the pedigree — unknown

Z : design matrix linking observations to breeding values — known

other components: same as sire model

OHP Picture 3



OHP Picture 4

Mixed Model Equations (MME) : Assume $var(e) = R$
$= I \cdot \sigma_e^2$

$$\begin{bmatrix} X^TX & X^TZ \\ Z^TX & Z^TZ + \underbrace{G^{-1}}_{?} \cdot \sigma_e^2 \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^Ty \\ Z^Ty \end{bmatrix}$$

$$var(u) = G = \begin{bmatrix} var(u_1) & cov(u_1, u_2) & \cdots \\ cov(u_2, u_1) & var(u_2) & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

$var(u_1) = (1+F_1) \cdot \sigma_u^2$ where F_1 is the inbreeding coefficient of animal 1

$var(u_2) = (1+F_2) \cdot \sigma_u^2$

$\vdots$

Chapter 6

if parents of animal are related

$cov(u_1, u_2) = ?$; if animal 1 and 2 are unrelated, then

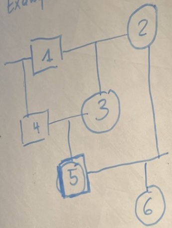
$cov(u_1, u_2) = \emptyset$

where relationship is defined by the pedigree (Stammbaum)

OHP Picture 5

Relationship between animals based on pedigree:

□ Example



$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

$$var(u) = \begin{bmatrix} var(u_1) & \overbrace{cov(u_1, u_2)}^{0} & \cdots \\ \vdots & & \\ \vdots & & \end{bmatrix}$$

Related means: animals have common ancestors

- Based on pedigree: animals 1 and 2 are not related

  $\Rightarrow cov(u_1, u_2) = 0$

- $cov(u_1, u_3)$: animal 3 is an offspring of parent 1

  $\Rightarrow$ decompose u_3 into: $\cancel{u_3 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_3}$

$$u_3 = \frac{1}{2} u_1 + \frac{1}{2} u_2 + m_3$$

OHP Picture 6

$$\mathrm{cov}(u_1,u_3) = \mathrm{cov}(u_1, \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + m_3)$$

$$= \mathrm{cov}(u_1, \tfrac{1}{2}u_1) + \mathrm{cov}(u_1, \tfrac{1}{2}u_2) + \mathrm{cov}(u_1, m_3)$$

$$= \tfrac{1}{2}\mathrm{cov}(u_1,u_1) + \tfrac{1}{2}\underbrace{\mathrm{cov}(u_1,u_2)}_{0} + \underbrace{\mathrm{cov}(u_1,m_3)}_{0}$$

$$= \tfrac{1}{2}\mathrm{var}(u_1) + \tfrac{1}{2}\cdot 0 + 0$$

$$= \tfrac{1}{2}(1+F_1)\cdot\sigma_u^2$$; F_1 corresponds to $\frac{1}{2}$ of relationship between parents of 1 and 2

$\Rightarrow F_1 = 0$

$$= \tfrac{1}{2}\sigma_u^2$$

$$\mathrm{cov}(u_1,u_4) = \ldots = \tfrac{1}{2}\sigma_u^2$$

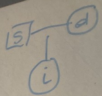
$$\mathrm{cov}(u_3,u_4) = \mathrm{cov}(\tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + m_3, \tfrac{1}{2}u_1 + m_4)$$

$$= \mathrm{cov}(\tfrac{1}{2}u_1, \tfrac{1}{2}u_1) + \mathrm{cov}(\tfrac{1}{2}u_2, \tfrac{1}{2}u_1)$$

$$= \tfrac{1}{4}\mathrm{cov}(u_1,u_1) + \tfrac{1}{4}\mathrm{cov}(u_2,u_1)$$

$$= \tfrac{1}{4}\sigma_u^2$$

OHP Picture 7

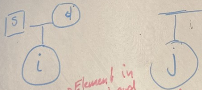


General:

$$G = var(u) = A \cdot \sigma_u^2$$

A: Numerator Relationship Matrix

(Additiv genetische Verwandtschaftsmatrix)



$$cov(u_i, u_j) = \underbrace{(A)_{ij}}_{\text{Element in row } i \text{ and column } j \text{ of matrix } A} \cdot \sigma_u^2 = \left[\frac{1}{2}(A_{sj}) + \frac{1}{2}(A_{dj})\right] \cdot \sigma_u^2$$

$$cov(u_1, u_3) = cov\left(u_1, \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_3\right)$$

$$= cov\left(u_1, \frac{1}{2}u_1\right) + cov\left(u_1, \frac{1}{2}u_2\right) + cov(u_1, m_3)$$

$$= \frac{1}{2}cov(u_1, u_1) + \frac{1}{2}\underbrace{cov(u_1, u_2)}_{0} + cov(u_1, m_3)$$

$$= \frac{1}{2}var(u_1) + \frac{1}{2} \cdot 0 + 0$$

$$= \frac{1}{2}(1 + F_1) \cdot \sigma_u^2$$

; F_1 corresponds to ½ of

OHP Picture 8

Numerator Relationship Matrix A

- Computation
  - diagonal elements: $(A)_{ii} = (1+F_i) = 1 + \frac{1}{2}(A)_{sd}$

    where s and d are known parents of i

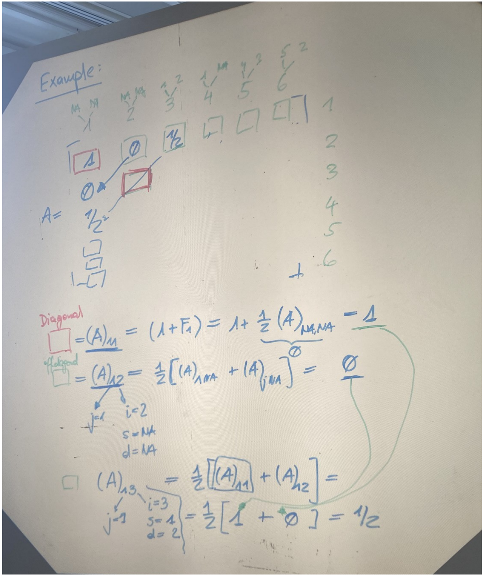
    and $(A)_{sd}$ stands for element in row s and column d of matrix A
  - offdiagonal: $(A)_{ji} = \frac{1}{2}(A)_{js} + \frac{1}{2}(A)_{jd}$

    where s and d are parents of i

$cov(u_j, u_i) = (A)_{ji} \cdot \sigma_u^2$; $u_i = \frac{1}{2}u_s + \frac{1}{2}u_d + m_i$

$$
\begin{aligned}
cov(u_j, u_i) &= cov\left(u_j, \tfrac{1}{2}u_s + \tfrac{1}{2}u_d + m_i\right) \\
&= cov\left(u_j, \tfrac{1}{2}u_s\right) + cov\left(u_j, \tfrac{1}{2}u_d\right) \\
&= \tfrac{1}{2}(A)_{js} \cdot \sigma_u^2 + \tfrac{1}{2}(A)_{jd} \cdot \sigma_u^2 \\
&= \underbrace{\left[\tfrac{1}{2}(A)_{js} + \tfrac{1}{2}(A)_{jd}\right]}_{(A)_{ji}} \cdot \sigma_u^2
\end{aligned}
$$

OHP Picture 9



OHP Picture 10

- So far we have seen: $var(u) = G = A \cdot \sigma_u^2$
- But for MME, we need $G^{-1} = A^{-1} \cdot \sigma_u^{-2}$
- In summary: BLUP animal models are only possible in real data sets because solutions can be obtained from MME and there is an efficient algorithm to directly compute A^{-1} without computing A.
- Direct construction of A^{-1} is based on the so-called LDL-decomposition of A

$A = L \cdot D \cdot L^T$ where L is lower-triangular and D is diagonal matrix

$$L = \begin{bmatrix} \ddots & 0 \\ & \ddots \end{bmatrix} ; \quad D = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$$

OHP Picture 11

Inverse of Matrix A!

□ $A^{-1} = (L^T)^{-1} \cdot D^{-1} \cdot L^{-1}$ why?

$A \cdot A^{-1} = I$ with $A = L \cdot D \cdot L^T$

$[L \cdot D \cdot L^T] \cdot (L^T)^{-1} \cdot D^{-1} \cdot L^{-1} =$

$L \cdot D \cdot \underbrace{L^T (L^T)^{-1}}_{I} \cdot D^{-1} \cdot L^{-1} = L \cdot \underbrace{D \cdot D^{-1}}_{I} \cdot L^{-1} = L \cdot L^{-1} = I$

□ Useful because L^{-1} and D^{-1} are easy to compute

□ Decomposition of breeding values: $u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + m_i$

$$\left.\begin{aligned} u_1 &= m_1 \\ u_2 &= m_2 \\ u_3 &= m_3 \\ u_4 &= \tfrac{1}{2} u_1 + \tfrac{1}{2} u_2 + m_4 \\ u_5 &= \tfrac{1}{2} u_3 + \tfrac{1}{2} u_2 + m_5 \end{aligned}\right\} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} =$$

$$u = P \cdot u + m$$

OHP Picture 12

Recursive Decomposition

- Simple: $u = P \cdot u + m$
- $u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + m_i$

  continue with u_s and u_d

  $u_s = \frac{1}{2} u_{ss} + \frac{1}{2} u_{ds} + m_s$

  $u_d = \frac{1}{2} u_{sd} + \frac{1}{2} u_{dd} + m_d$

- Example:

  $u_1 = \underline{m_1}$

  $u_2 = m_2$

  $u_3 = m_3$

  $u_4 = \frac{1}{2} u_1 + \frac{1}{2} u_2 + m_4 = \frac{1}{2} m_1 + \frac{1}{2} m_2 + m_4$

  $u_5 = \frac{1}{2} u_3 + \frac{1}{2} u_2 + m_5 = \frac{1}{2} m_3 + \frac{1}{2} m_2 + m_5$

$\Rightarrow u = L \cdot m$