OHP Picture 1

Recap: □ Covariances between related individuals

□ Reason: MME contain the matrix G
where $G = A \cdot \sigma_u^2 = var(u)$

numerator relationship matrix

□ More precisely: G^{-1} is required for the coefficient matrix of MME

□ Since $G = A \cdot \sigma_u^2 \Leftrightarrow G^{-1} = A^{-1} \cdot \sigma_u^{-2}$
$\Rightarrow A^{-1}$ required. (In practice A^{-1} has dimensions 10^7 rows $\times$ 10^7 columns)

□ Idea: - Construct A^{-1} without computing A
- Construction is based on LDL-decomposition of A, where $A = L \cdot D \cdot L^T$
where L is a lower-triangular matrix and D is a diagonal matrix
- L and D are "easy" to invert

OHP Picture 2

□ Simple decomposition of u_i of animal i:

$$u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + m_i$$

where u_s : breeding value of parents } of i

u_d : " " " of

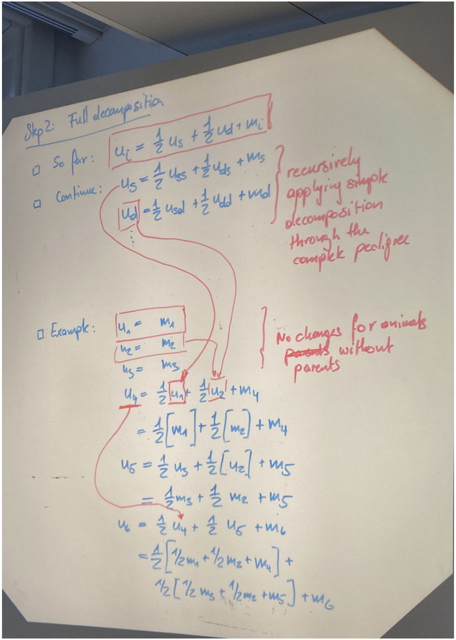
m_i : ~~me~~ mendelian sampling term.

$$\begin{bmatrix} u_1 = \\ u_2 = \\ u_3 = \\ u_4 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + \\ u_5 = \frac{1}{2}u_3 + \frac{1}{2}u_2 + \\ u_6 = \frac{1}{2}u_4 + \frac{1}{2}u_5 + \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}$$

$$u = P \cdot u + m$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + m$$

OHP Picture 3



OHP Picture 4

$$u_6 = \frac{1}{2}\left[\tfrac{1}{2} m_1 + \tfrac{1}{2} m_2 + m_4\right] + \tfrac{1}{2}\left[\tfrac{1}{2} m_3 + \tfrac{1}{2} m_2 + m_5\right] + m_6$$

$$= \tfrac{1}{4} m_1 + \tfrac{1}{4} m_2 + \tfrac{1}{2} m_4 + \tfrac{1}{4} m_3 + \tfrac{1}{4} m_2 + \tfrac{1}{2} m_5 + m_6$$

$$= \tfrac{1}{4} m_1 + \tfrac{1}{2} m_2 + \tfrac{1}{4} m_3 + \tfrac{1}{2} m_4 + \tfrac{1}{2} m_5 + m_6$$

- Summary:

$\to 0 \Rightarrow L$ is lower-triangular

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \tfrac{1}{2} & \tfrac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 1 & 0 \\ \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{2} & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}}_{m}$$

OHP Picture 5

Properties of Matrix L:

$$\underset{u}{\begin{bmatrix} u_1 \\ \vdots \\ \boxed{u_4} \\ \vdots \\ u_6 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ \boxed{0.5} & \boxed{0.5} & 0 & 1 & 0 & 0 \\ 0 & & & & & \\ \boxed{0.25} & & & & & \end{bmatrix}} \underset{m}{\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}}$$

$(L)_{61}$ ⇒ fraction of m_1 in u_6, m_1 comes from either $\overbrace{\text{father 4}}^{(L)_{41}}$ or $\underbrace{\text{mother 5}}_{(L)_{51}}$

$u_4 = \boxed{\tfrac{1}{2}} m_1 + \tfrac{1}{2} m_2 + m_4$

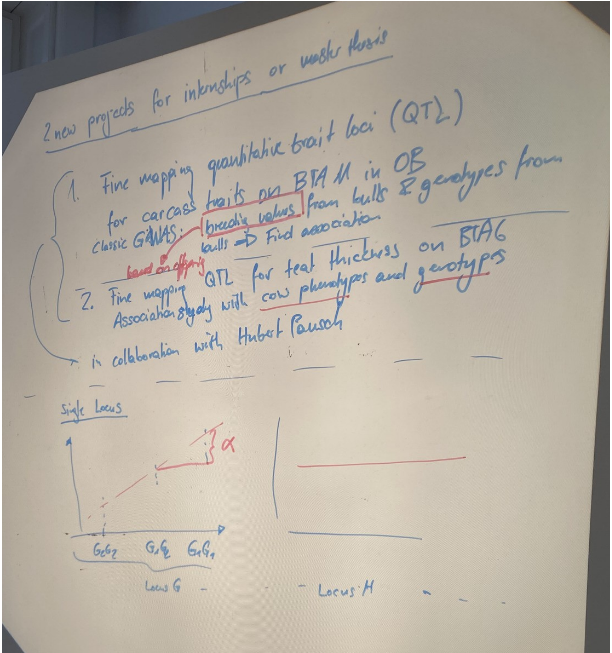
$\underbrace{(L)_{41}}_{\text{fraction of } m_1 \text{ in } u_4} = 0.5 = \tfrac{1}{2}\left[(L)_{11} + (L)_{21}\right]$ ⇒ $\tfrac{1}{2}(L)_{51} + \tfrac{1}{2}(L)_{41}$

where animals 1 and 2 are parents of animal 4

→ In general: $(L)_{ij}$ which is the element in row i and column j ($i > j$ meaning the lower diagonal of L)

$(L)_{ij} = \tfrac{1}{2}(L)_{sj} + \tfrac{1}{2}(L)_{dj}$ where s and d are parents of i

OHP Picture 6



OHP Picture 8

Matrix L:

- $u = L \cdot m$
- $(L)_{ij} = \frac{1}{2}(L)_{sj} + \frac{1}{2}(L)_{dj}$

Decomposition for var (u):

- Because $u = L \cdot m$

$$var(u) = var(L \cdot m)$$
$$= L \cdot var(m) \cdot L^T$$

- The vector m contains random mendelian sampling terms (m_i) for animal i:

Full sibs i and j with parents s and d

$$u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + \boxed{m_i}$$
$$u_j = \frac{1}{2} u_s + \frac{1}{2} u_d + \boxed{m_j}$$

in general $u_i \neq u_j$ because i and j did receive the same sample of random allels from parents

$m_i \neq m_j$

But m_i and m_j are indepdent $\Leftrightarrow cov(m_i, m_j) = \emptyset$

OHP Picture 9

- $\mathrm{cov}(m_i, m_j) = 0$ for all m_i and m_j if $i \neq j$

$$\Rightarrow \mathrm{var}(m) = \begin{bmatrix} \mathrm{var}(m_1) & \mathrm{cov}(m_1, m_2) & \mathrm{cov}(m_1, m_3) & \cdots \\ \mathrm{cov}(m_2, m_1) & \mathrm{var}(m_2) & \cdots & \\ \vdots & & & \end{bmatrix}$$

$= 0$

$\Rightarrow \mathrm{var}(m)$ will be a diagonal matrix

- $\mathrm{var}(m_1), \mathrm{var}(m_2), \ldots, \mathrm{var}(m_i)$?
- Using $u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + m_i$

$$\begin{aligned} \mathrm{var}(u_i) &= \mathrm{var}\left[\tfrac{1}{2} u_s + \tfrac{1}{2} u_d + m_i\right] \\ &= \mathrm{var}(\tfrac{1}{2} u_s) + \mathrm{var}(\tfrac{1}{2} u_d) + \mathrm{var}(m_i) \\ &\quad + 2\,\mathrm{cov}(\tfrac{1}{2} u_s, \tfrac{1}{2} u_d) \\ &\quad + \underbrace{2\,\mathrm{cov}(\tfrac{1}{2} u_s, m_i) + 2\,\mathrm{cov}(\tfrac{1}{2} u_d, m_i)}_{=0} \\ &= \tfrac{1}{4}\mathrm{var}(u_s) + \tfrac{1}{4}\mathrm{var}(u_d) + \tfrac{1}{2}\mathrm{cov}(u_s, u_d) \\ &\quad + \mathrm{var}(m_i) \end{aligned}$$

OHP Picture 10

$$\mathrm{var}(u_i) = \frac{1}{4}\mathrm{var}(u_s) + \frac{1}{4}\mathrm{var}(u_d) + \frac{1}{2}\mathrm{cov}(u_s, u_d) + \underline{\mathrm{var}(m_i)}$$

- Definition of $G = A \cdot \sigma_u^2$, we know that

$$\begin{cases} \mathrm{var}(u_s) = (1+F_s)\,\sigma_u^2 \\ \mathrm{var}(u_d) = (1+F_d)\,\sigma_u^2 \\ \mathrm{cov}(u_d, u_s) = (A)_{sd}\,\sigma_u^2 = 2F_i \cdot \sigma_u^2 \end{cases}$$

where F_k is the inbreeding coefficient of animal k

- Solve for $(\mathrm{var}(m_i))$:

$$\underline{\mathrm{var}(m_i)} = \mathrm{var}(u_i) - \frac{1}{4}\mathrm{var}(u_s) - \frac{1}{4}\mathrm{var}(u_d) - \frac{1}{2}\mathrm{cov}(u_s, u_d)$$

$$= (1+F_i)\sigma_u^2 - \frac{1}{4}(1+F_s)\sigma_u^2 - \frac{1}{4}(1+F_d)\sigma_u^2 - \frac{1}{2}(2F_i\sigma_u^2)$$

$$= \underline{\left(\frac{1}{2} - \frac{1}{4}(F_s+F_d)\right)\sigma_u^2}$$

for know parents s and d of animal i

OHP Picture 11

$$\mathrm{var}(u) = \mathrm{var}(L \cdot m)$$
$$= L \cdot \mathrm{var}(m) \cdot L^T$$

with $\mathrm{var}(m)$ being a diagonal matrix D with diagonal elements

$$\sigma_u^2 \cdot (D)_{ii} = \mathrm{var}(m_i) = \begin{cases} \left(\frac{1}{2} - \frac{1}{4}(F_s + F_d)\right)\sigma_u^2 & (1) \\ (3/4 - 1/4\, F_s)\,\sigma_u^2 & (2) \\ \sigma_u^2 & (3) \end{cases}$$

where (1) : both parents known
(2) : only one parent s is known
(3) : no parents known

$$\Rightarrow \mathrm{var}(u) = L \cdot D\sigma_u^2 L^T \quad \text{where } \sigma_u^2 D = \mathrm{var}(m)$$
$$= \underbrace{L \cdot D \cdot L^T}_{} \cdot \sigma_u^2$$
$$= A \cdot \sigma_u^2$$

OHP Picture 12

Inverse A^{-1} of A:

- $A = L \cdot D \cdot L^T$

  $\Rightarrow A^{-1} = (L^T)^{-1} \cdot D^{-1} \cdot L^{-1}$

  D^{-1} diagonal

- Because D is diagonal $\Rightarrow D^{-1}$ is diagonal with diagonal elements $(D^{-1})_{ii} = 1/(D)_{ii}$

$$D = \begin{bmatrix} var(m_1) & 0 & \cdots & 0 \\ 0 & var(m_2) & \cdots & \\ & & \ddots & \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/var(m_1) & 0 & \cdots & 0 \\ 0 & 1/var(m_2) & \cdots & 0 \\ & & & \end{bmatrix}$$

- L^{-1} ?

OHP Picture 13

Matrix L^{-1}:

- Decompositions:

$$u = \boxed{P} \cdot u + m$$
$$u = L \cdot m$$

both decompositions of the same vector u

$$\Rightarrow u - P \cdot u + m = L \cdot m$$

$$\Rightarrow P \cdot u = L \cdot m - m = (L - I)m$$

- Solve both decompositions for m

$$\rightarrow m = u - Pu = (I - P) \cdot u$$
$$m = L^{-1} \cdot u$$

both equations hold for the same m

$$\Rightarrow (I-P) \cdot u = L^{-1} \cdot u$$

$$\Downarrow$$

$$L^{-1} = (I - P)$$

where I is the identity matrix and P is the matrix from the simple decomposition

OHP Picture 14