Numerator Relationship Matrix

Peter von Rohr

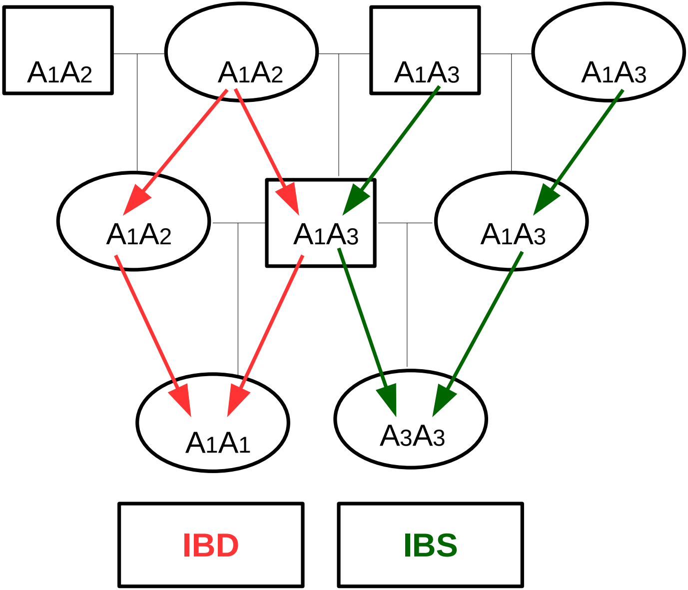
2021-11-12

Similarity Between Individuals

At the genetic level there are two different kinds of similarity

1. Identity by descent (IBD)
2. Identity by state

IBD versus IBS



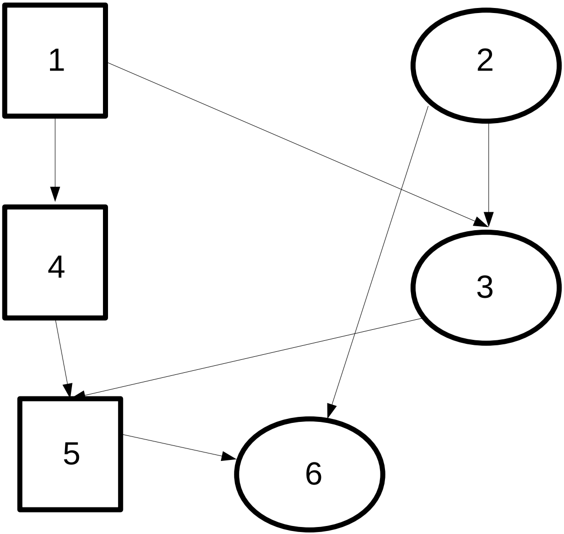
Numerator Relationship Matrix

- probability of IBD alleles in two individuals: **coancestry** or **coefficient of kinship**
- additive genetic relationship between two individuals is twice their coancestry
- matrix containing all additive genetic relationships in a population is called **numerator relationship matrix** (A)
- A is symmetric and contains on
  - diagonal: $(A)_{ii} = (1 + F_i)$
  - off-diagonal: $(A)_{ij} = cov(u_i, u_j)/\sigma_u^2$ (with $i \neq j$)

Recursive Computation of A

- If both parents s and d of animal i are known then
  - the diagonal element $(A)_{ii}$ corresponds to: $(A)_{ii} = 1 + F_i = 1 + \frac{1}{2}(A)_{sd}$ and
  - the offdiagonal element $(A)_{ji}$ is computed as: $(A)_{ji} = \frac{1}{2}((A)_{js} + (A)_{jd})$
  - because A is symmetric $(A)_{ji} = (A)_{ij}$
- If only one parent s is known and assumed unrelated to the mate
  - $(A)_{ii} = 1$
  - $(A)_{ij} = (A)_{ji} = \frac{1}{2}((A)_{js}$
- If both parents are unknown
  - $(A)_{ii} = 1$
  - $(A)_{ij} = (A)_{ji} = 0$

Example



Tabular Representation of Pedigree

Table 1: Example Pedigree To Compute Additive Genetic Relationship Matrix

Calf	Sire	Dam
3	1	2
4	1	NA
5	4	3
6	5	2

Stepwise Computation of A

- Start by extending pedigree with animals that do not have parents
- Order animals, such that parents before progeny

Animal	Sire	Dam
1	NA	NA
2	NA	NA
3	1	2
4	1	NA
5	4	3
6	5	2

Initialize With Empty Matrix A

- Dimensions of A: number of rows and number of columns equal to the number of animals
- Our example: 6×6

$$A = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

First Diagonal Element

- Compute first element $(A)_{11} = 1 + F_1$
- Animal 1 has both parents unknown $\rightarrow F_1 = 0$

$$A = \begin{bmatrix} 1.00 & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Off-diagonal Elements

- Assume animal i has parents s and d
- $(A)_{ji} = \frac{1}{2}((A)_{js} + (A)_{jd})$

First Row of A

$$A = \begin{bmatrix} 1.00 & 0.00 & 0.50 & 0.50 & 0.50 & 0.25 \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

Use Symmetry of A

- Copy first row into first column

$$A = \begin{bmatrix} 1.00 & 0.00 & 0.50 & 0.50 & 0.50 & 0.25 \\ 0.00 & & & & & \\ 0.50 & & & & & \\ 0.50 & & & & & \\ 0.50 & & & & & \\ 0.25 & & & & & \end{bmatrix}$$

Remaining Elements of A

- Continue with rows and columns 2 to 6 using the same recipe

Final Result

$$A = \begin{bmatrix} 1.0000 & 0.0000 & 0.5000 & 0.5000 & 0.5000 & 0.2500 \\ 0.0000 & 1.0000 & 0.5000 & 0.0000 & 0.2500 & 0.6250 \\ 0.5000 & 0.5000 & 1.0000 & 0.2500 & 0.6250 & 0.5625 \\ 0.5000 & 0.0000 & 0.2500 & 1.0000 & 0.6250 & 0.3125 \\ 0.5000 & 0.2500 & 0.6250 & 0.6250 & 1.1250 & 0.6875 \\ 0.2500 & 0.6250 & 0.5625 & 0.3125 & 0.6875 & 1.1250 \end{bmatrix}$$

The Inverse Numerator Relationship Matrix

- Recap: Henderson's mixed model equations depend on four matrices

1. Design matrix X for the fixed effects
2. Design matrix Z for the random effects
3. The inverse variance-covariance matrix R^{-1} for the residuals e and
4. The inverse variance-covariance matrix G^{-1} for the random breeding values a.

Animal Model

- Breeding values of all individuals as random effects
- Variance-Covariance matrix G corresponds to variance-covariance matrix of breeding values

$$G = A * \sigma_u^2$$

- We need: G^{-1}

$$G^{-1} = A^{-1} * \frac{1}{\sigma_u^2}$$

Need For Efficient Computation of $A{-}1$

- In practical livestock breeding evaluations A is very large
- Dimensions of A can be $10^7 \times 10^7$
- Explicit general inversion not possible
- Special structure of A^{-1} leads to efficient computation