# Inverse Numerator Relationship Matrix

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## Problem 1: Inverse Numerator Relationship Matrix

During the lecture the method of computing the inverse numerator relationship matrix $A^{-1}$ directly was introduced. The computation is based on the LDL-decomposition. As a result, we can write

$$A^{-1} = (L^T)^{-1} \cdot D^{-1} \cdot L^{-1}$$

where $L^{-1} = I - P$, and $D^{-1}$ is a diagonal matrix with $(D^{-1})_{ii} * \sigma_u^{-2} = var(m_i)^{-1}$.

### Tasks

- Use the example pedigree given below and compute the matrices $L^{-1}$ and $D^{-1}$ to compute $A^{-1}$
- Verify your result using the function `getAinv()` from package pedigreemm.

### Pedigree

```
nr_animal <- 6
tbl_pedigree <- tibble::tibble(Calf = c(1:nr_animal),
                               Sire = c(NA, NA, NA, 1 ,3, 4),
                               Dam = c(NA, NA, NA, 2, 2, 5))
tbl_pedigree
```

```
## # A tibble: 6 x 3
##    Calf  Sire   Dam
##   <int> <dbl> <dbl>
## 1     1    NA    NA
## 2     2    NA    NA
## 3     3    NA    NA
## 4     4     1     2
## 5     5     3     2
## 6     6     4     5
```

### Solution

The matrix $P$ comes from the simple decomposition and can be constructed using the pedigree.

```
P = matrix(0, nrow = nr_animal, ncol = nr_animal)
for (i in 1:nr_animal){
  s <- tbl_pedigree$Sire[i]
  d <- tbl_pedigree$Dam[i]
  if (!is.na(s)){
    P[i,s] <- 0.5
  }
```

```
  if(!is.na(d)){
    P[i,d] <- 0.5
  }
}
P
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]  0.0  0.0  0.0  0.0  0.0    0
## [2,]  0.0  0.0  0.0  0.0  0.0    0
## [3,]  0.0  0.0  0.0  0.0  0.0    0
## [4,]  0.5  0.5  0.0  0.0  0.0    0
## [5,]  0.0  0.5  0.5  0.0  0.0    0
## [6,]  0.0  0.0  0.0  0.5  0.5    0
```

With that the matrix $L^{-1}$ is

```
I <- diag(1, nrow = nr_animal, ncol = nr_animal)
Linv <- I - P
Linv
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]  1.0  0.0  0.0  0.0  0.0    0
## [2,]  0.0  1.0  0.0  0.0  0.0    0
## [3,]  0.0  0.0  1.0  0.0  0.0    0
## [4,] -0.5 -0.5  0.0  1.0  0.0    0
## [5,]  0.0 -0.5 -0.5  0.0  1.0    0
## [6,]  0.0  0.0  0.0 -0.5 -0.5    1
```

The matrix $D$ is obtained from package pedigreemm

```
ped <- pedigreemm::pedigree(sire = tbl_pedigree$Sire,
                            dam  = tbl_pedigree$Dam,
                            label = as.character(1:nr_animal))
D <- pedigreemm::Dmat(ped = ped)
Dinv <- diag(1/D, nrow = nr_animal, ncol = nr_animal)
Dinv
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]    1    0    0    0    0    0
## [2,]    0    1    0    0    0    0
## [3,]    0    0    1    0    0    0
## [4,]    0    0    0    2    0    0
## [5,]    0    0    0    0    2    0
## [6,]    0    0    0    0    0    2
```

The inverse numerator relationship matrix is

```
Ainv <- t(Linv) %*% Dinv %*% Linv
Ainv
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6]
## [1,]  1.5  0.5  0.0 -1.0  0.0    0
## [2,]  0.5  2.0  0.5 -1.0 -1.0    0
## [3,]  0.0  0.5  1.5  0.0 -1.0    0
## [4,] -1.0 -1.0  0.0  2.5  0.5   -1
## [5,]  0.0 -1.0 -1.0  0.5  2.5   -1
## [6,]  0.0  0.0  0.0 -1.0 -1.0    2
```

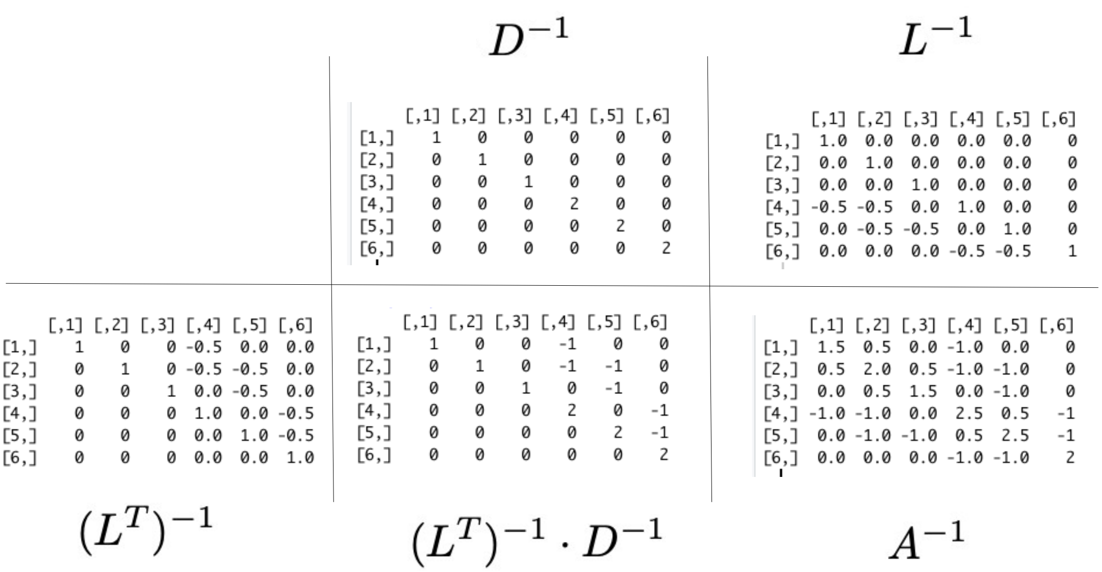
### Verification

```r
pedigreemm::getAInv(ped = ped)
```

```
## 6 x 6 Matrix of class "dgeMatrix"
##      1    2    3    4    5  6
## 1  1.5  0.5  0.0 -1.0  0.0  0
## 2  0.5  2.0  0.5 -1.0 -1.0  0
## 3  0.0  0.5  1.5  0.0 -1.0  0
## 4 -1.0 -1.0  0.0  2.5  0.5 -1
## 5  0.0 -1.0 -1.0  0.5  2.5 -1
## 6  0.0  0.0  0.0 -1.0 -1.0  2
```

## Problem 2: Rules

The following diagram helps to illustrate the rules for constructing $A^{-1}$




### Tasks

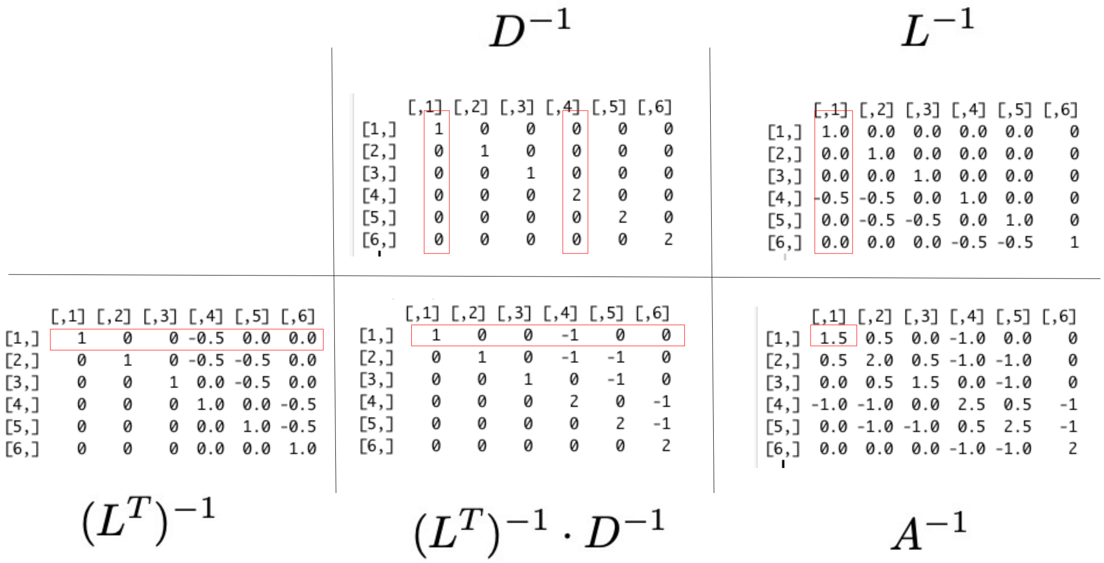
- Go through the list of animals in the pedigree and write down the contributions that are made to the different elements of matrix $A^{-1}$
- Based on the different contributions, try to come up with some general rules

### Solution

In what follows, we use the following convention $\delta_i = (D^{-1})_{ii}$.

**Animal 1**

We start with animal 1.




Animal 1 has no parents and therefore the diagonal element $\delta_1 = (D^{-1})_{11}$ of matrix $D^{-1}$ is $\delta_1 = 1$. By looking at the red boxes, we can see that $\delta_1$ is added as a contribution to $(A^{-1})_{11}$. So far we are still missing a contribution of 0.5 to the element $(A^{-1})_{11}$. Again by inspecting the red boxes in the above diagram, we can see that this contribution corresponds to $\delta_4/4$ which comes from offspring 4 of parent 1. Hence diagonal elements of $(A^{-1})_{ss}$ corresponding to parents $s$ of offsprint $i$ receive $\delta_i/4$ as contribution. More details on that is obtained when inspecting the contributions of animal 4. Animals 2 and 3 do not have parents and are therefore analogous to animal 1.

**Animal 4**

Animal 4 has parents 1 and 2.

$D^{-1}$

| | [,1] | [,2] | [,3] | [,4] | [,5] | [,6] |
|---|---|---|---|---|---|---|
| [1,] | 1 | 0 | 0 | 0 | 0 | 0 |
| [2,] | 0 | 1 | 0 | 0 | 0 | 0 |
| [3,] | 0 | 0 | 1 | 0 | 0 | 0 |
| [4,] | 0 | 0 | 0 | 2 | 0 | 0 |
| [5,] | 0 | 0 | 0 | 0 | 2 | 0 |
| [6,] | 0 | 0 | 0 | 0 | 0 | 2 |

$L^{-1}$

| | [,1] | [,2] | [,3] | [,4] | [,5] | [,6] |
|---|---|---|---|---|---|---|
| [1,] | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0 |
| [2,] | 0.0 | 1.0 | 0.0 | 0.0 | 0.0 | 0 |
| [3,] | 0.0 | 0.0 | 1.0 | 0.0 | 0.0 | 0 |
| [4,] | -0.5 | -0.5 | 0.0 | 1.0 | 0.0 | 0 |
| [5,] | 0.0 | -0.5 | -0.5 | 0.0 | 1.0 | 0 |
| [6,] | 0.0 | 0.0 | 0.0 | -0.5 | -0.5 | 1 |

| | [,1] | [,2] | [,3] | [,4] | [,5] | [,6] |
|---|---|---|---|---|---|---|
| [1,] | 1 | 0 | 0 | -0.5 | 0.0 | 0.0 |
| [2,] | 0 | 1 | 0 | -0.5 | -0.5 | 0.0 |
| [3,] | 0 | 0 | 1 | 0.0 | -0.5 | 0.0 |
| [4,] | 0 | 0 | 0 | 1.0 | 0.0 | -0.5 |
| [5,] | 0 | 0 | 0 | 0.0 | 1.0 | -0.5 |
| [6,] | 0 | 0 | 0 | 0.0 | 0.0 | 1.0 |

$(L^T)^{-1}$

| | [,1] | [,2] | [,3] | [,4] | [,5] | [,6] |
|---|---|---|---|---|---|---|
| [1,] | 1 | 0 | 0 | -1 | 0 | 0 |
| [2,] | 0 | 1 | 0 | -1 | -1 | 0 |
| [3,] | 0 | 0 | 1 | 0 | -1 | 0 |
| [4,] | 0 | 0 | 0 | 2 | 0 | -1 |
| [5,] | 0 | 0 | 0 | 0 | 2 | -1 |
| [6,] | 0 | 0 | 0 | 0 | 0 | 2 |

$(L^T)^{-1} \cdot D^{-1}$

| | [,1] | [,2] | [,3] | [,4] | [,5] | [,6] |
|---|---|---|---|---|---|---|
| [1,] | 1.5 | 0.5 | 0.0 | -1.0 | 0.0 | 0 |
| [2,] | 0.5 | 2.0 | 0.5 | -1.0 | -1.0 | 0 |
| [3,] | 0.0 | 0.5 | 1.5 | 0.0 | -1.0 | 0 |
| [4,] | -1.0 | -1.0 | 0.0 | 2.5 | 0.5 | -1 |
| [5,] | 0.0 | -1.0 | -1.0 | 0.5 | 2.5 | -1 |
| [6,] | 0.0 | 0.0 | 0.0 | -1.0 | -1.0 | 2 |

$A^{-1}$

The red boxes in the above diagram show that for animal 4 there is a contribution of $\delta_4$ to the diagonal. Then there are contributions of $\delta_4/4$ for the elements $(A^{-1})_{11}$, $(A^{-1})_{22}$, $(A^{-1})_{12}$ and $(A^{-1})_{21}$. Furthermore there are negative contributions of $\delta_4/2$ to $(A^{-1})_{41}$, $(A^{-1})_{14}$, $(A^{-1})_{24}$ and $(A^{-1})_{42}$.

### General Rules

From this the general rules which were first published by Henderson can be deduced as

- Both Parents Known
  - add $\delta_i$ to the diagonal-element $(i,i)$
  - add $-\delta_i/2$ to off-diagonal elements $(s,i)$, $(i,s)$, $(d,i)$ and $(i,d)$
  - add $\delta_i/4$ to elements $(s,s)$, $(d,d)$, $(s,d)$, $(d,s)$
- Only One Parent Known
  - add $\delta_i$ to diagonal-element $(i,i)$
  - add $-\delta_i/2$ to off-diagonal elements $(s,i)$, $(i,s)$
  - add $\delta_i/4$ to element $(s,s)$
- Both Parents Unknown
  - add $\delta_i$ to diagonal-element $(i,i)$