Inverse Numerator Relationship Matrix

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2022-11-11

Recap: 2022-11-18

* Introduction into numerator relationship matrix A
* Computation of elements of A
* MME require A^{-1}, the inverse of A
* For our examples in the course, A^{-1} by pedigreemm::getAinv(), only possible for small datasets.

==> Find a strategy to efficiently construct A^{-1} directly from the pedigree, without first constructing A.

Structure of A^{-1}

- Look at a simple example of A and A^{-1}

Table 1: Pedigree Used To Compute Inverse Numerator Relationship Matrix

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2

Founder animals = Animals without known parents

Numerator Relationship Matrix A

$$A = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & 0.5000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.5000 & 0.5000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.5000 \\ 0.5000 & 0.5000 & 0.0000 & 1.0000 & 0.2500 \\ 0.0000 & 0.5000 & 0.5000 & 0.2500 & 1.0000 \end{bmatrix}$$

In R: pedigreemm:getA()

Inverse Numerator Relationship Matrix A^{-1}

$$A^{-1} = \begin{bmatrix} 1.5000 & 0.5000 & 0.0000 & -1.0000 & 0.0000 \\ 0.5000 & 2.0000 & 0.5000 & -1.0000 & -1.0000 \\ 0.0000 & 0.5000 & 1.5000 & 0.0000 & -1.0000 \\ -1.0000 & -1.0000 & 0.0000 & 2.0000 & 0.0000 \\ 0.0000 & -1.0000 & -1.0000 & 0.0000 & 2.0000 \end{bmatrix}$$

Conclusions

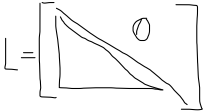
- A^{-1} has simpler structure than A itself
- Non-zero elements only at positions of parent-progeny and parent-mate positions
- Parent-mate positions are positive, parent-progeny are negative

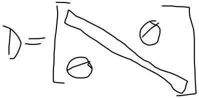
Henderson's Rules

- Based on `LDL`-decomposition of A

$$A = L * D * L^{T}$$

where	L	Lower triangular matrix
	D	Diagonal matrix





- Why?
  - matrices L and D can be inverted directly, we 'll see how . . .
  - construct $A^{-1} = (L^{T})^{-1} * D^{-1} * L^{-1}$

Given A can be decomposed into the product:

A = L * D * L^{T}, then this can be used to construct the inverse A^{-1},

namely, it follows that

A^{-1} = (L^T)^{-1} * D^~~{-1} * L^{-1}~~

this is important, because, inverses L^{-1} and D^{-1} are simpler to compute

Why is A^{-1} = (L^T)^{-1} * D^{-1} * L^{-1} ?

Because A^{-1} is defined as the matrix that satisfies A^{-1}*A = I

Check:
((L^T)^{-1} * D^{-1} * L^{-1})*L * D * L^{T} = (L^T)^{-1} * D^{-1} * I * D * L^{T}
= (L^T)^{-1} * D^{-1} * D * L^{T}=(L^T)^{-1} * L^{T} = I

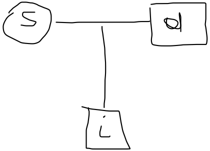
Example

$$L = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.5 & 0.5 & 0.0 & 1.0 \end{bmatrix} \rightarrow \text{lower triangular}$$

$$D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

→ Verify that $A = L * D * \boxed{L^T} \rightarrow$ upper triangular

Decomposition of True Breeding Value



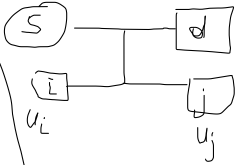
- True breeding value (u_i) of animal i

$$u_i = \underline{\frac{1}{2}u_s + \frac{1}{2}u_d} + \underline{m_i}$$

Mendelian sampling deviation

- Do that for all animals in pedigree

$$\underbrace{u_i \neq u_j}_{m_i \neq m_j} \Rightarrow \begin{array}{l} u_i = \ldots \\ u_j = \frac{1}{2}u_s + \frac{1}{2}u_d \\ \quad + m_j \end{array}$$



Decomposition for Example

$$\boxed{\begin{aligned} u_1 &= m_1 \\ u_2 &= m_2 \\ u_3 &= m_3 \end{aligned}} \longrightarrow \text{founders}$$

$$\begin{aligned} u_4 &= \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_4 \\ u_5 &= \frac{1}{2}u_3 + \frac{1}{2}u_2 + m_5 \end{aligned}$$

Matrix Vector Notation

- Define vectors u and m as
- Coefficients of u_s and u_d into matrix P

$$
u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix},\ m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix},\ P = \begin{array}{c|ccccc} \text{Calf} & 1 & 2 & 3 & 4 & 5 \rightarrow \text{Parents} \\ \hline 1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ \rightarrow 4 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 5 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 \end{array}
$$

- Result: Decomposition of true breeding values

$$
u = P \cdot u + m
$$

Recursive Decomposition

Decomposition so far was for the true breeding value of animal i into breeding values of parents s and d. Continuing this decomposition with breeding values of parents s and d, leads to the following

- True breeding values of s and d can be decomposed into

$$\begin{aligned} u_s &= \frac{1}{2}\overset{\text{sire of s}}{u_{ss}} + \frac{1}{2}\overset{\text{dam of s}}{u_{ds}} + m_s \\ u_d &= \frac{1}{2}u_{sd} + \frac{1}{2}u_{dd} + m_d \end{aligned}$$

grand parents of i

where	ss	sire of s
	ds	dam of s
	sd	sire of d
	dd	dam of d

Example

- Add animal 6 with parents 4 and 5 to our example pedigree

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2
6	4	5

First Step Of Decomposition

$$u_1 = m_1$$

$$u_2 = m_2$$

$$u_3 = m_3$$

$$u_4 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_4$$

$$u_5 = \frac{1}{2}u_3 + \frac{1}{2}u_2 + m_5$$

$$u_6 = \frac{1}{2}u_4 + \frac{1}{2}u_5 + m_6$$

$u = P \cdot u + m$

Decompose Parents

$$u_4 = \frac{1}{2}\boxed{u_1} + \frac{1}{2}u_2 + m_4$$

$$u_1 = m_1$$

$$u_2 = m_2$$

$$u_3 = m_3$$

$$\rightarrow u_4 = \frac{1}{2}m_1 + \frac{1}{2}m_2 + m_4$$

$$u_5 = \frac{1}{2}m_3 + \frac{1}{2}m_2 + m_5$$

$$u_6 = \frac{1}{2}\underbrace{\left(\frac{1}{2}(u_1 + u_2) + m_4\right)}_{u_4} + \frac{1}{2}\underbrace{\left(\frac{1}{2}(u_3 + u_2) + m_5\right)}_{u_5} + m_6$$

$$= \frac{1}{4}(u_1 + u_2) + \frac{1}{2}m_4 + \frac{1}{4}(u_3 + u_2) + \frac{1}{2}m_5 + m_6$$

Decompose Grand Parents

- Only animal 6 has true breeding values for grand parents

$$
\begin{aligned}
u_6 &= \frac{1}{4}(\overset{u_1 = m_1}{\overset{\uparrow}{u_1}} + u_2) + \frac{1}{2}m_4 + \frac{1}{4}(u_3 + u_2) + \frac{1}{2}m_5 + m_6 \\
&= \frac{1}{4}m_1 + \frac{1}{4}m_2 + \frac{1}{4}m_3 + \frac{1}{4}m_2 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6 \\
&= \frac{1}{4}m_1 + \frac{1}{2}m_2 + \frac{1}{4}m_3 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6
\end{aligned}
$$

Summary

$$
\begin{aligned}
u_1 &= m_1 \\
u_2 &= m_2 \\
u_3 &= m_3 \\
u_4 &= \frac{1}{2}m_1 + \frac{1}{2}m_2 + m_4 \\
u_5 &= \frac{1}{2}m_3 + \frac{1}{2}m_2 + m_5 \\
u_6 &= \frac{1}{4}m_1 + \frac{1}{2}m_2 + \frac{1}{4}m_3 + \frac{1}{2}m_4 + \frac{1}{2}m_5 + m_6
\end{aligned}
$$

only dependent on m-terms

Matrix-Vector Notation

- Use vectors u and m again

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}, \quad m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}, \quad L = \begin{array}{c|cccccc} {}_{\text{Calf}}\backslash^{\text{Parent}} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 2 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 3 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 4 \rightarrow & \boxed{0.50} & \boxed{0.50} & 0.00 & \boxed{1.00} & 0.00 & 0.00 \\ 5 & 0.00 & \boxed{0.50} & \boxed{0.50} & 0.00 & 1.00 & 0.00 \\ \rightarrow 6 & \boxed{0.25} & \underbrace{\boxed{0.50}}_{1/4+1/4} & \boxed{0.25} & \boxed{0.50} & \boxed{0.50} & 1.00 \end{array}$$

- Result of recursive decomposition of u_i

$$u = L \cdot m$$

Property of L

- Meaning of Element $(L)_{ij}$ of Matrix L:

$$\begin{array}{ccccc} u & = & L & * & m \\ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \boxed{u_4} \\ u_5 \\ u_6 \end{bmatrix} & = & \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ \boxed{0.50} & 0.50 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.50 & 0.50 & 0.00 & 1.00 & 0.00 \\ \boxed{0.25} & 0.50 & 0.25 & 0.50 & 0.50 & 1.00 \end{bmatrix} & * & \begin{bmatrix} \boxed{m_1} \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix} \end{array}$$

$(L)_{61}$

calf

$(L)_{41}$

ancestor

For t generations back between animal and ancestor, the coefficient in Matrix L is: $\left(\frac{1}{2}\right)^t$

Property of L II

Mendelian Sampling term of
ancestor j of animal i

- Element $(L)_{ij}$ $(i > j)$ is the proportion of m_j in $\underline{u_i}$
- Given: i has parents s and d
- m_j can only come from u_s and u_d, because $u_i = 1/2 u_s + 1/2 u_d + m_i$
- The proportion of m_j in u_i is half the proportion of m_j in u_s and half the proportion of m_j in u_d

$$\rightarrow L_{ij} = \frac{1}{2} L_{sj} + \frac{1}{2} L_{dj}$$

If animals i and j are not related, that means, if j is not an ancestor of i ==> L_{ij} = 0, (L)_{51} = 0

Example

- L_{41}, L_{62}

$$\boxed{L_{41}} = \tfrac{1}{2}\left(L_{11} + L_{21}\right)$$

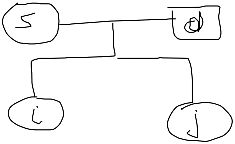
$$\boxed{L_{62}}$$

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2
6	4	5

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \boxed{u_4} \\ u_5 \\ \boxed{u_6} \end{bmatrix} = \begin{bmatrix} \boxed{1.00} & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ \boxed{0.00} & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ \boxed{0.50} & \boxed{0.50} & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & \boxed{0.50} & 0.50 & 0.00 & 1.00 & 0.00 \\ 0.25 & \boxed{0.50} & 0.25 & 0.50 & 0.50 & 1.00 \end{bmatrix} * \begin{bmatrix} \boxed{m_1} \\ \boxed{m_2} \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{bmatrix}$$

I

Variance From Recursive Decomposition



Full recursive decomposition of breeding values u: u = L * m

Variance-Covariance matrix of breeding values: $G = A * \sigma_u^2$

constant

$$\overbrace{var(u)} = var(\overbrace{L}^{\text{constant}} \cdot m)$$

$$= L \cdot \underbrace{var(\underbrace{m}_{\text{vector of mendelian sampling terms}})}_{\text{Variance-Covariance matrix of all components of the vector m}} \cdot L^T$$

Variance-Covariance matrix of all components of the vector m

vector of mendelian sampling terms

where $var(m)$ is the variance-covariance matrix of all components in vector m.

Matings between parents of i and j are independent events

- covariances of components m_i, $\underbrace{\overbrace{cov(m_i, m_j) = 0} \text{ for } i \neq j}$
- $\underline{var(m_i)}$ computed as shown below

==> var(m) is a diagonal matrix. On the diagonal of var(m), we can find the variance (var(m_i)) for the mendelian sampling component (m_i) of animal i.

What is the Variance $var(m_i)$

- Decomposition of $var(u_i)$ using $\underbrace{u_i = 1/2u_s + 1/2u_d + \overset{\circledast}{m_i}}_{\text{simple decomposition}}$

$$\overbrace{var(u_i)}^{\text{scalar (just a number)}} = \underset{\circledast}{var}(\underbrace{1/2u_s + 1/2u_d + m_i})$$

$$= var(1/2u_s) + var(1/2u_d) + \frac{1}{2} * cov(u_s, u_d) + var(m_i)$$

$$= 1/4var(\underline{u_s}) + 1/4\underline{var(u_d)} + \frac{1}{2} * \underline{cov(u_s, u_d)} + \underbrace{var(m_i)}_{?}$$

- From the definition of $G = A * \sigma_u^2$

i-th diagonal element of G

i-th diagonal element of A

Solve for unknown

$$\overbrace{var(u_i)} = \overbrace{(1 + F_i)}\sigma_u^2$$

$$var(u_s) = (1 + F_s)\sigma_u^2$$

$$var(u_d) = (1 + F_d)\sigma_u^2$$

$$\underbrace{cov(u_s, u_d)} = (A)_{sd}\sigma_u^2 = 2F_i\sigma_u^2$$

$$F_i = \frac{1}{2} * (A)_{sd}$$

Element in row s and column d of G

Variance of Mendelian Sampling Terms

- What is $var(m_i)$?
- Solve equation for $var(u_i)$ for $var(m_i)$

$$var(m_i) = \underline{var(u_i)} - 1/4var(u_s) - 1/4var(u_d) - 2 * cov(u_s, u_d)$$

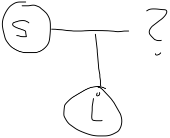
- Insert definitions from ~~A~~ G

$$\underline{var(m_i)} = \overbrace{(1 + F_i)\sigma_u^2} - 1/4(1 + F_s)\sigma_u^2 - 1/4(1 + F_d)\sigma_u^2 - \frac{1}{2} * 2 * F_i\sigma_u^2$$
$$= \left(\frac{1}{2} - \frac{1}{4}(F_s + F_d)\right)\boxed{\underline{\sigma_u^2}}$$

- True, for both parents s and d of animal i are known

Unknown Parents

- Only parent s of animal i is known

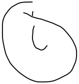


$$u_i = \frac{1}{2}u_s + \underline{m_i}$$

$$var(m_i) = \left(1 - \frac{1}{4}(1 + F_s)\right)\sigma_u^2$$

$$= \underline{\left(\frac{3}{4} - \frac{1}{4}F_s\right)\boxed{\sigma_u^2}}$$

- Both $\underbrace{\text{parents are unknown}}_{\text{founder}}$



$$u_i = m_i$$

$$var(m_i) = \underline{1 \cdot \sigma_u^2}$$

Using var(m_i) to build up the diagonal matrix: var(m)

$$\begin{bmatrix} \overset{\sigma_u^2}{var(m_1)} & & \emptyset \\ \emptyset & var(m_2) & \\ \emptyset & & \ddots \\ & & & \underset{\left(1/2 - (F_s + F_d)1/4\right)\sigma_u^2}{var(m_q)} \end{bmatrix}$$

Result

- variance-covariance matrix $var(m)$ can be written as $D * \sigma_u^2$ where D is diagnoal

$$\begin{bmatrix} 1\cdot\sigma_u^2 & & 0 \\ & 1\cdot\sigma_u^2 & \\ 0 & & \left(\frac{1}{2}-\frac{1}{4}(F_S+F_D)\right)\sigma_u^2 \end{bmatrix} = D\cdot\sigma_u^2$$

$$G = A\cdot\sigma_u^2$$

$$\rightarrow \overbrace{var(u)}^{G} = L \cdot \overbrace{var(m)} \cdot L^T$$

$$= L \cdot \overbrace{D * \underline{\sigma_u^2}} \cdot L^T \rightarrow \text{scalar}$$

$$= \underbrace{L \cdot D \cdot L^T} * \underline{\sigma_u^2}$$

$$= A\underline{\sigma_u^2} = G$$

$$\rightarrow A = L \cdot D \cdot L^T$$

$$D = \begin{bmatrix} var(m_i) & & 0 \\ & \ddots & \\ 0 & & var(m_i) \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \end{bmatrix}$$

Inverse of A Based on L and D

$$A^{-1} = \underbrace{(L^T)^{-1}}_{(L^{-1})^T} * D^{-1} * L^{-1} = \underbrace{(L^{-1})^T * D^{-1} * L^{-1}}$$

- Matrix A was decomposed into $A = L \cdot D \cdot L^T$
- Get A^{-1} as $A^{-1} = (L^T)^{-1} D^{-1} L^{-1}$
- D^{-1} is diagonal again with elements

$$(D^{-1})_{ii} = 1/(D)_{ii}$$

$$D = \begin{bmatrix} (D)_{11} & & 0 \\ & (D)_{22} & \\ & & \ddots \\ & & & (D)_{qq} \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1/(D)_{11} & & 0 \\ & 1/(D)_{22} & \\ 0 & & \ddots \\ & & & 1/(D)_{qq} \end{bmatrix}$$

$$\overbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \end{matrix}}^{I}$$

$$L^{-1} * L = I$$

$$(L^{-1} * L)^T = I^T = I$$

$$\underbrace{L^T}_{M} * \underbrace{(L^{-1})^T}_{N} = I$$

$$M * \underbrace{N}_{M^{-1}} = I$$

$$M^{-1} = (L^T)^{-1} \implies \boxed{(L^{-1})^T = (L^T)^{-1}}$$

Inverse of L $\Longrightarrow L^{-1}$?

- Compute m based on the two decompositions of u

$$\overbrace{u = \boxed{P}\cdot u + \underline{m}}^{\text{simple}} \quad \text{and} \quad \overbrace{u = L\cdot \underline{m}}^{\text{full-recursive}}$$

$\cdot(-1)$

pre-multiply w. L^{-1}

$$L^{-1}u = \underbrace{L^{-1}\cdot L}_{I} m$$

- Solve both for m and set them equal

$$m = u - P\cdot u = \underbrace{(I-P)}\cdot u \quad \text{and} \quad \overbrace{m = \underbrace{L^{-1}}\cdot u}$$

$$\underbrace{(I-P)}\cdot \underline{u} = \underbrace{L^{-1}}\cdot \underline{u}$$

$=$

and

$$L^{-1} = I - \underline{P}$$

So far:

$$\left.\begin{array}{l} D^{-1} \\ L^{-1} = I - P \\ (L^{-1})^T \end{array}\right\}$$

$$A^{-1} = (L^T)^{-1} * D^{-1} * L^{-1} = \overbrace{(L^{-1})^T * D^{-1} * L^{-1}}$$

Example

Calf	Sire	Dam
1	NA	NA
2	NA	NA
3	NA	NA
4	1	2
5	3	2

Matrix D^{-1}

- Because D is diagonal

founders

$\frac{var(m_1)}{\sigma_u^2}$

$var(m_4) = \left[\frac{1}{2} - \frac{1}{4}\overbrace{(F_s + F_d)}^{\emptyset}\right] \cdot \sigma_u^2$

$$D = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \end{bmatrix}$$

- We get D^{-1} as

$1/1$

$\frac{1}{1/2}$

$$D^{-1} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

Matrix L^{-1}

- Use $L^{-1} = I - P$
- Matrix P from simple decomposition

$$P = \begin{array}{c|ccccc|}
 & 1 & 2 & 3 & & \\
 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
4 & \underline{0.5} & \underline{0.5} & 0.0 & 0.0 & 0.0 \\
5 & 0.0 & \underline{0.5} & \underline{0.5} & 0.0 & 0.0
\end{array}$$

- Therefore

$$L^{-1} = I - P = \begin{bmatrix}
1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\
-\underline{0.5} & \underline{-0.5} & 0.0 & 1.0 & 0.0 \\
0.0 & \underline{-0.5} & \underline{-0.5} & 0.0 & 1.0
\end{bmatrix}$$

Decomposition of A^{-1} I

$$A^{-1} = (L^{-1})^T \cdot D^{-1} \cdot L^{-1}$$

$$\overbrace{(L^{-1})^T \cdot D^{-1}}^{*}$$

$$\overbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 & -0.5 & -0.5 \\ 0.0 & 0.0 & 1.0 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}}^{(L^{-1})^T} \cdot \overbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}}^{D^{-1}}$$

$$= \underbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}}_{*}$$

Decomposition of A^{-1} II

$$A^{-1} = (L^{-1})^T \cdot D^{-1} \cdot L^{-1}$$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix} \cdot \overbrace{\begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.5 & -0.5 & 0.0 & 1.0 \end{bmatrix}}^{L^{-1}}$$

$$= \underbrace{\begin{bmatrix} 1.5 & 0.5 & 0.0 & -1.0 & 0.0 \\ 0.5 & 2.0 & 0.5 & -1.0 & -1.0 \\ 0.0 & 0.5 & 1.5 & 0.0 & -1.0 \\ -1.0 & -1.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & -1.0 & -1.0 & 0.0 & 2.0 \end{bmatrix}}_{A^{-1}}$$

Decomposition of A^{-1} III

D^{-1}

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

L^{-1}

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -0.5 & -0.5 & 0.0 & 1.0 & 0.0 \\ 0.0 & -0.5 & -0.5 & 0.0 & 1.0 \end{bmatrix}$$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & -0.5 & 0.0 \\ 0.0 & 1.0 & 0.0 & -0.5 & -0.5 \\ 0.0 & 0.0 & 1.0 & 0.0 & -0.5 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$(L^{-1})^T$

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 \end{bmatrix}$$

$(L^{-1})^T \cdot D^{-1}$

$1 + \frac{1}{2}$ $\quad$ $1 + \frac{1}{2} + \frac{1}{2}$

$$\begin{array}{c} \\ 2 \\ 3 \\ 4 \\ \\ \end{array} \begin{bmatrix} 1.5 & 0.5 & 0.0 & -1.0 & 0.0 \\ 0.5 & 2.0 & 0.5 & -1.0 & -1.0 \\ 0.0 & 0.5 & 1.5 & 0.0 & -1.0 \\ -1.0 & -1.0 & 0.0 & 2.0 & 0.0 \\ 0.0 & -1.0 & -1.0 & 0.0 & 2.0 \end{bmatrix}$$

A^{-1}

Henderson's Rules

$$var(m_i) = \left(1/2 - 1/4(\underset{\downarrow}{F_s} + \underset{\downarrow}{F_d})\right)\sigma_u^2$$

- Both Parents Known
  - add 2 to the diagonal-element (i, i)
  - add -1 to off-diagonal elements (s, i), (i, s), (d, i) and (i, d)
  - add $\frac{1}{2}$ to elements (s, s), (d, d), (s, d), (d, s)
- Only One Parent Known
  - add $\frac{4}{3}$ to diagonal-element (i, i)
  - add $-\frac{2}{3}$ to off-diagonal elements (s, i), (i, s)
  - add $\frac{1}{3}$ to element (s, s)
- Both Parents Unknown
  - add 1 to diagonal-element (i, i)
- Valid without inbreeding

For a general pedigree with inbreeding, we have to know the inbreeding coefficients of all animals. Inbreeding coefficients can be obtained from the diagonal elements of A. But for large pedigree, we cannot compute all diagonal elements of A.