OHP Picture 1

①

Recap: 2023-10-20

- New Model: Infinitesimal model

infinite number of loci in the genome

$$y_{ij} = \mu_i + u_i + e_{ij}^*$$

unknown part of environment, plus dominance, epistasis

breeding value of animal i

Known environment (herd, season, age, ..)

(Later: Genomic selection: polygenic model

large but finite number of loci)

- Decomposition of u_i into parent average plus mendelian sampling term (m_i)

$$u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + m_i$$

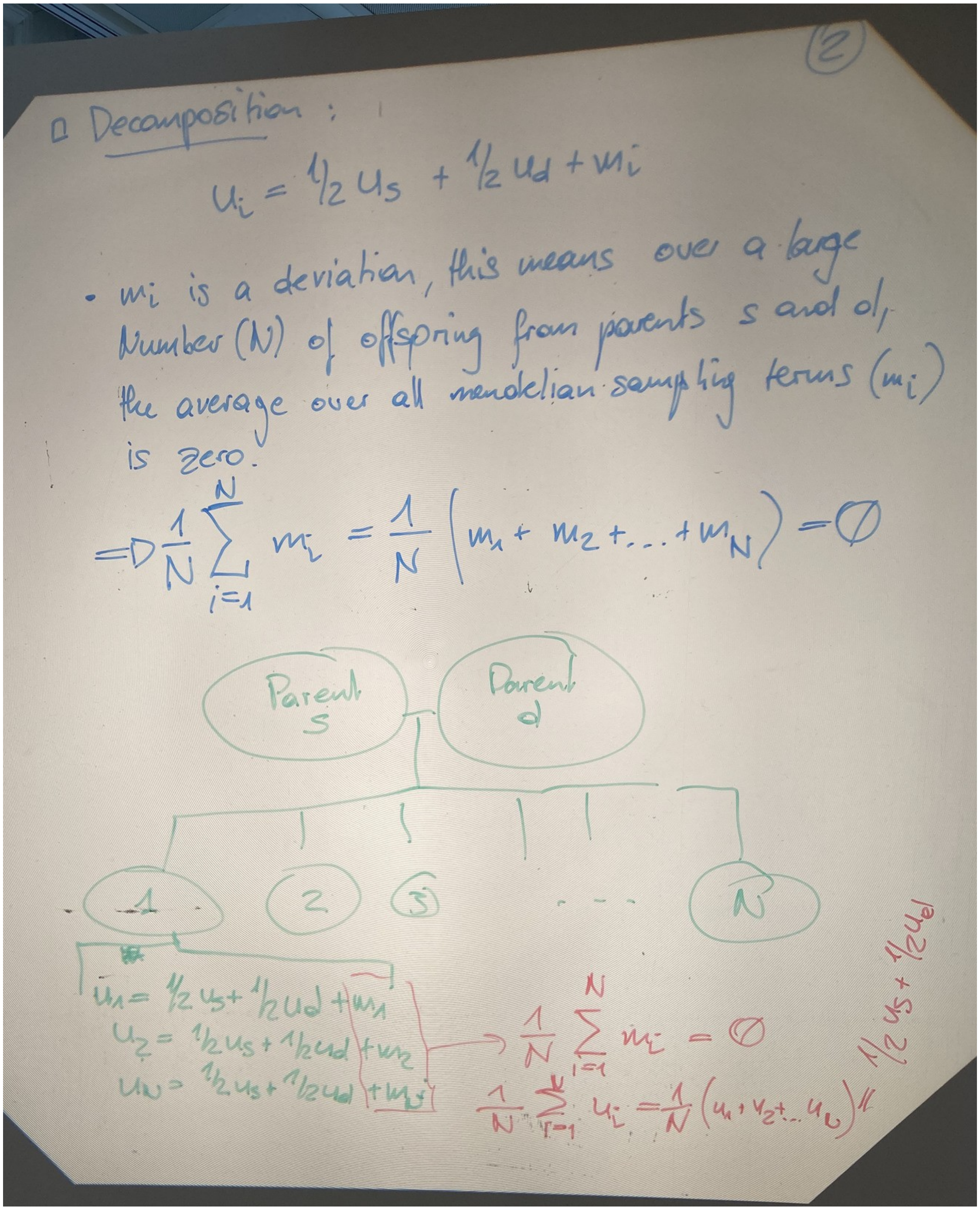
breeding value of animal d

breeding value of s

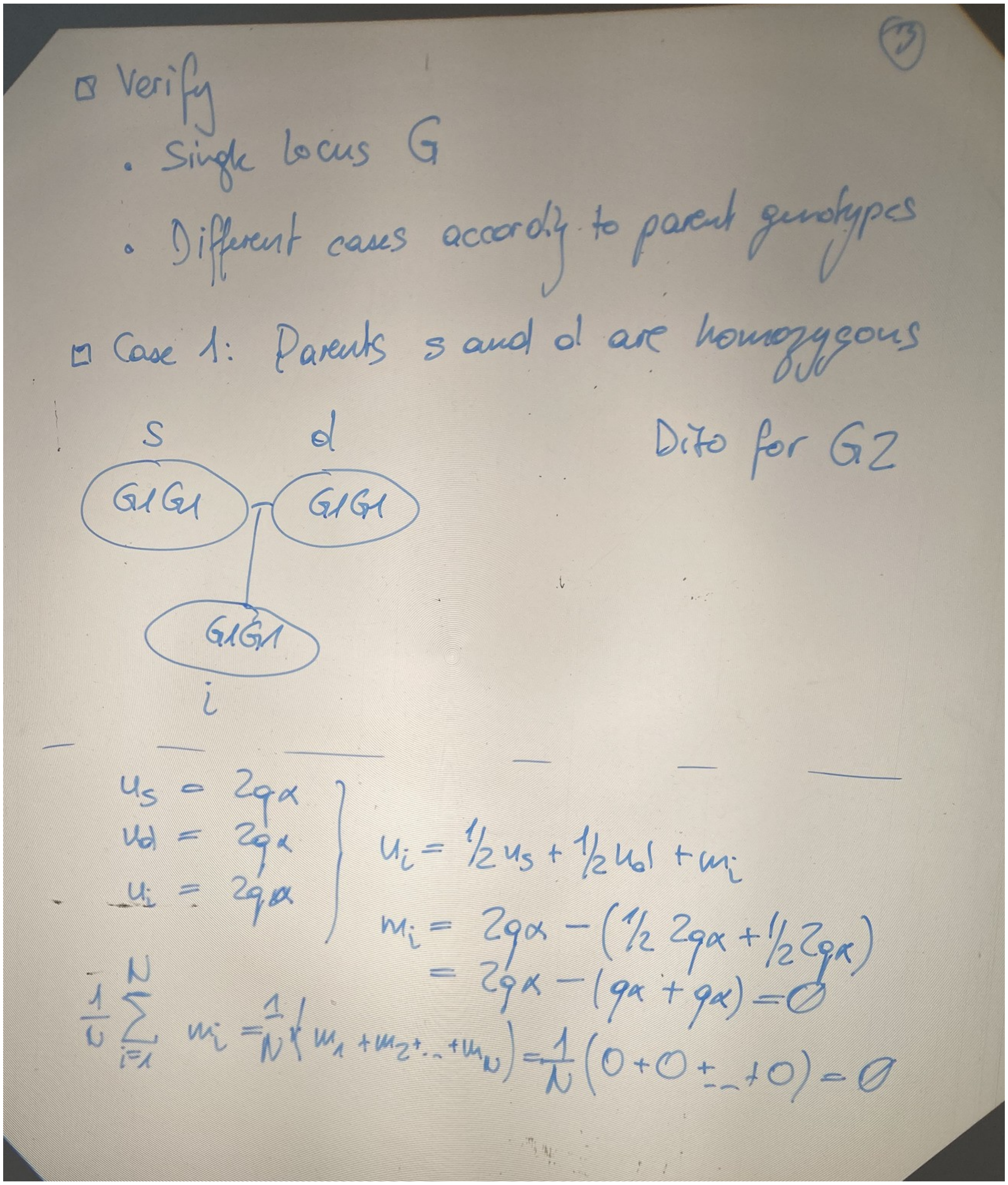
breeding value of animal i

Animals s and d are parents of i

OHP Picture 2



OHP Picture 3



(13)

☐ Verify

- Single locus G
- Different cases according to parent genotypes

☐ Case 1: Parents s and d are homozygous

Dito for G2

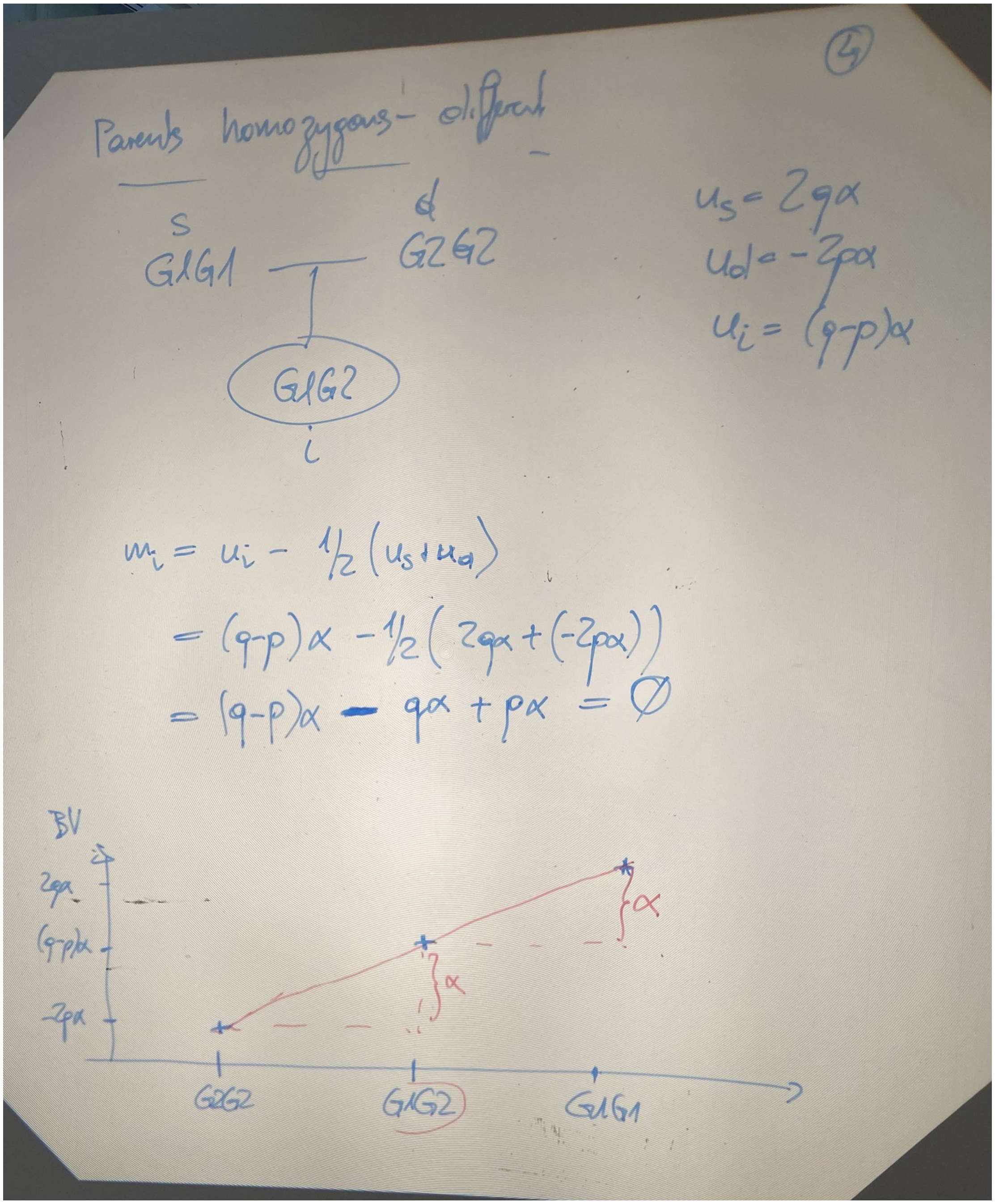
$$u_s = 2q\alpha,\quad u_d = 2q\alpha,\quad u_i = 2q\alpha$$

$$u_i = \tfrac{1}{2} u_s + \tfrac{1}{2} u_d + m_i$$

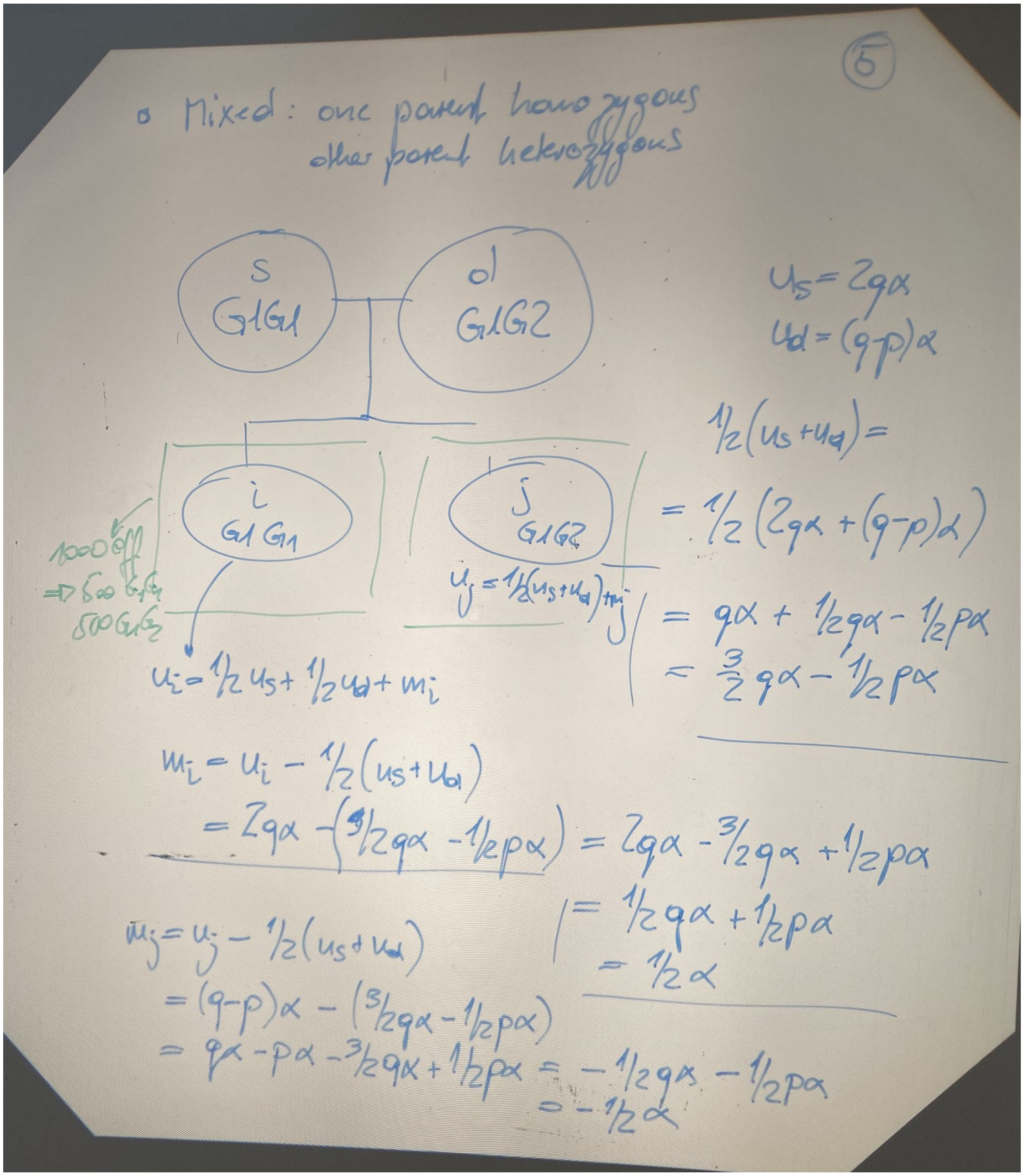
$$m_i = 2q\alpha - \left(\tfrac{1}{2} 2q\alpha + \tfrac{1}{2} 2q\alpha\right) = 2q\alpha - (q\alpha + q\alpha) = 0$$

$$\frac{1}{N}\sum_{i=1}^{N} m_i = \frac{1}{N}(m_1 + m_2 + \dots + m_N) = \frac{1}{N}(0 + 0 + \dots + 0) = 0$$

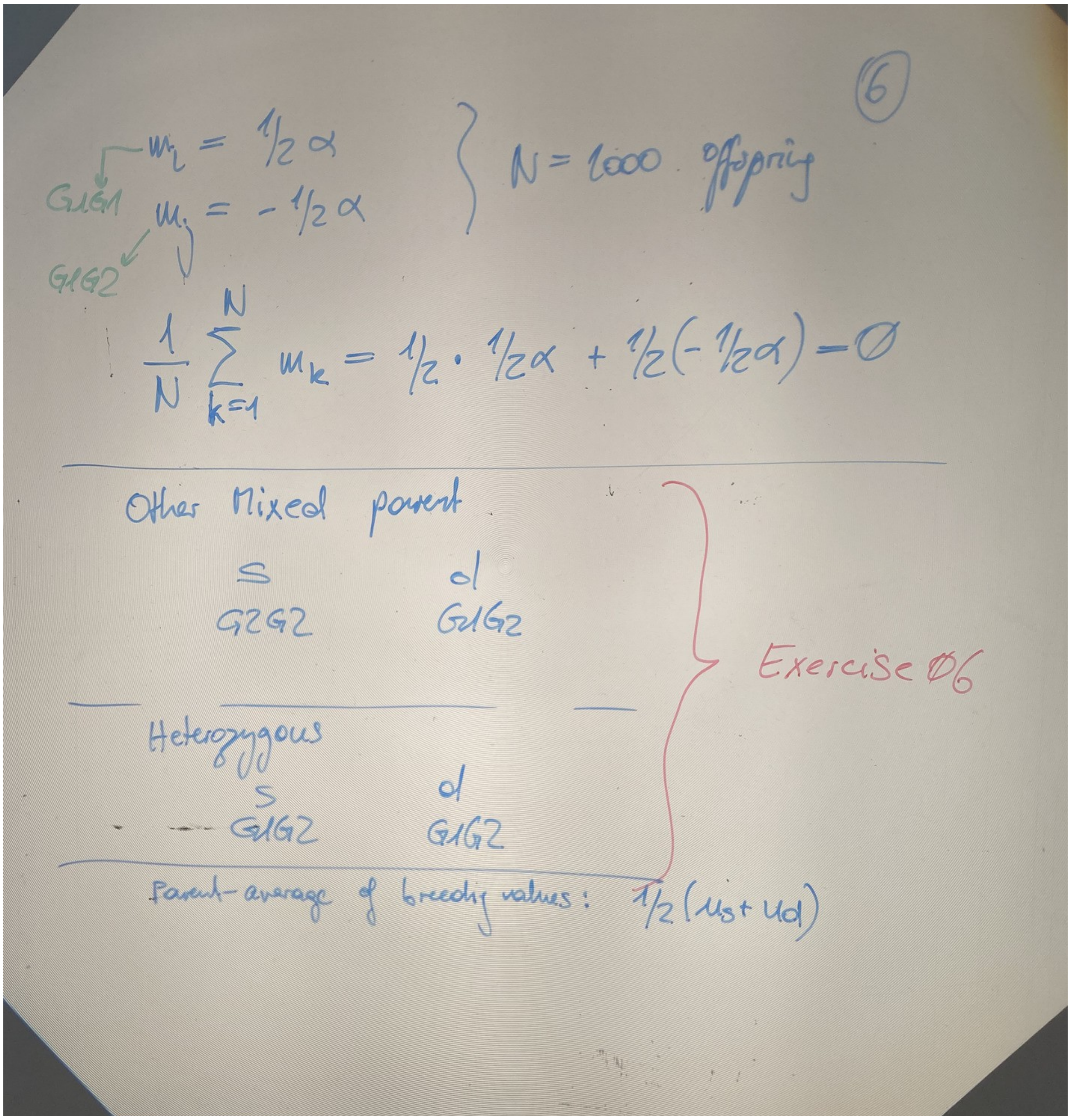
OHP Picture 4



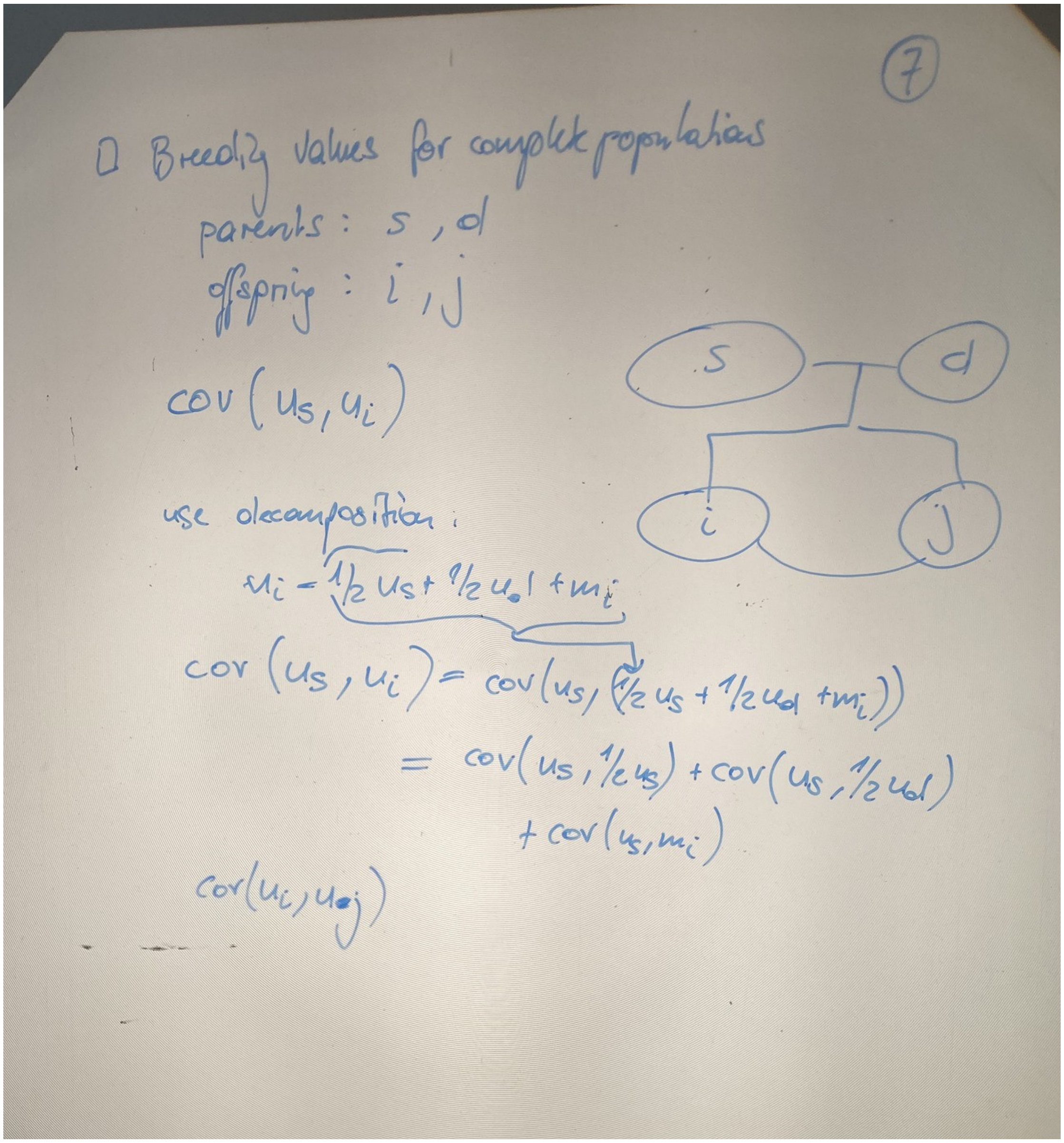
OHP Picture 5



OHP Picture 6



OHP Picture 7



OHP Picture 8

(8)

- Infinitesimal Model: For animal i

$$y_{ij} = \mu_i + u_i + e_{ij}^{*}$$

; u_i (and e_{ij}^{*}) unknown,

Goal: prediction for u_i based on y_{ij}

- Data set:

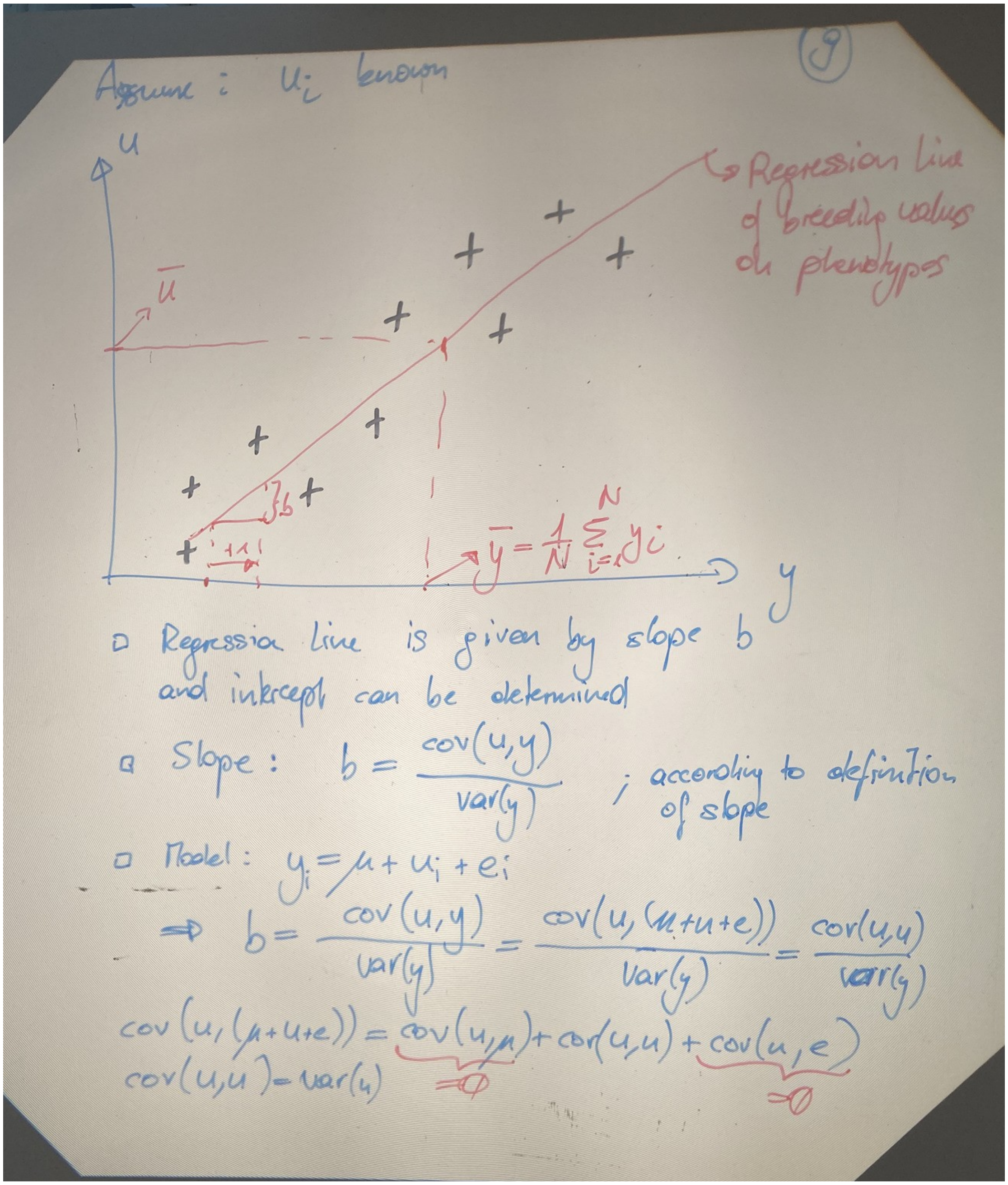
Animal	y (weight in kg)
1	$y_1 = 250$
2	$y_2 = 310$
⋮	⋮
N	$y_N = 293$

One observation per animal

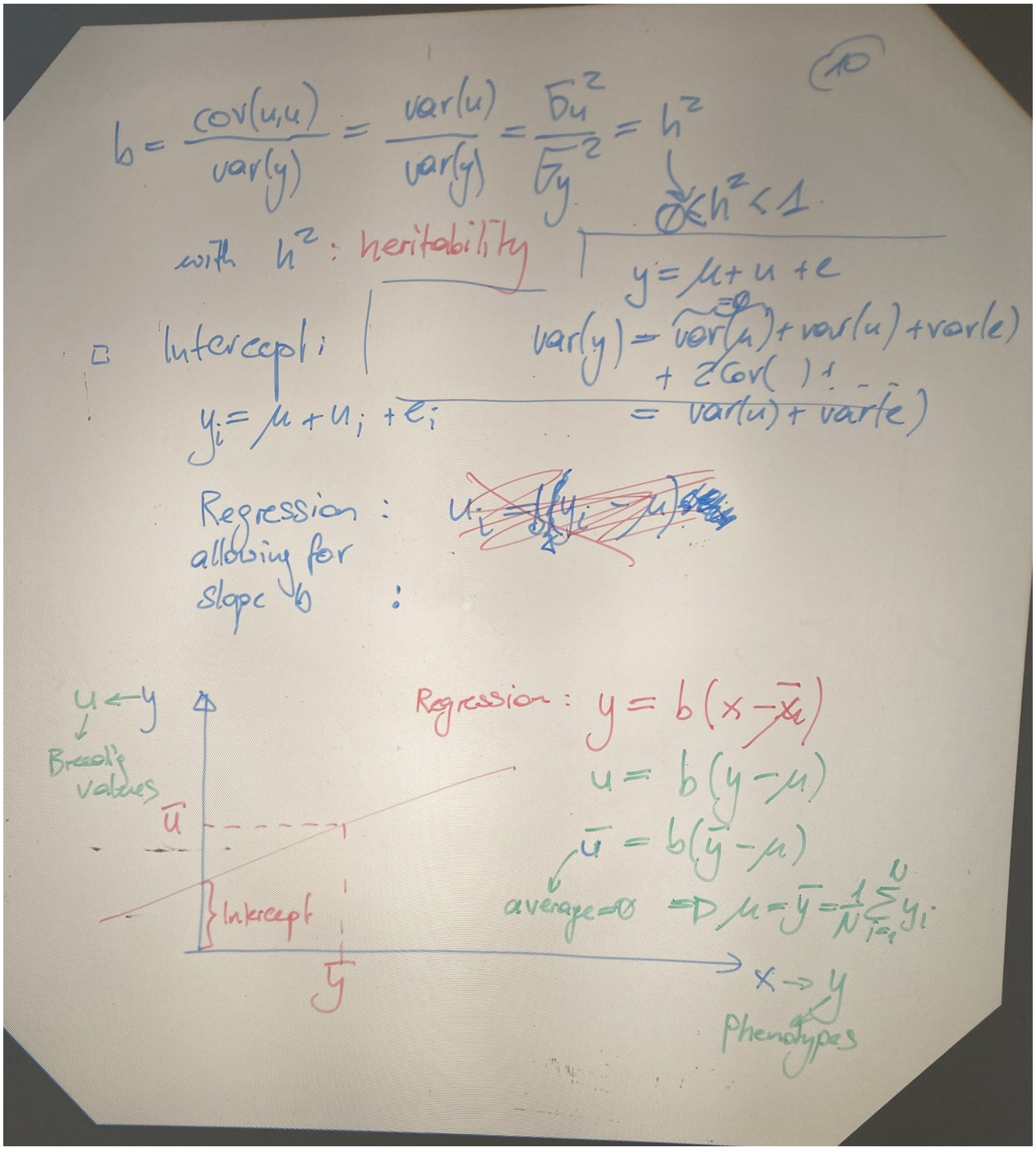
weight after 1 year

- Assume:
  Use selection of parents such that weight after 1 year is improved.
- Predict breeding values based on available information
  Call predicted breeding value for animal i: $\hat{u}_i$

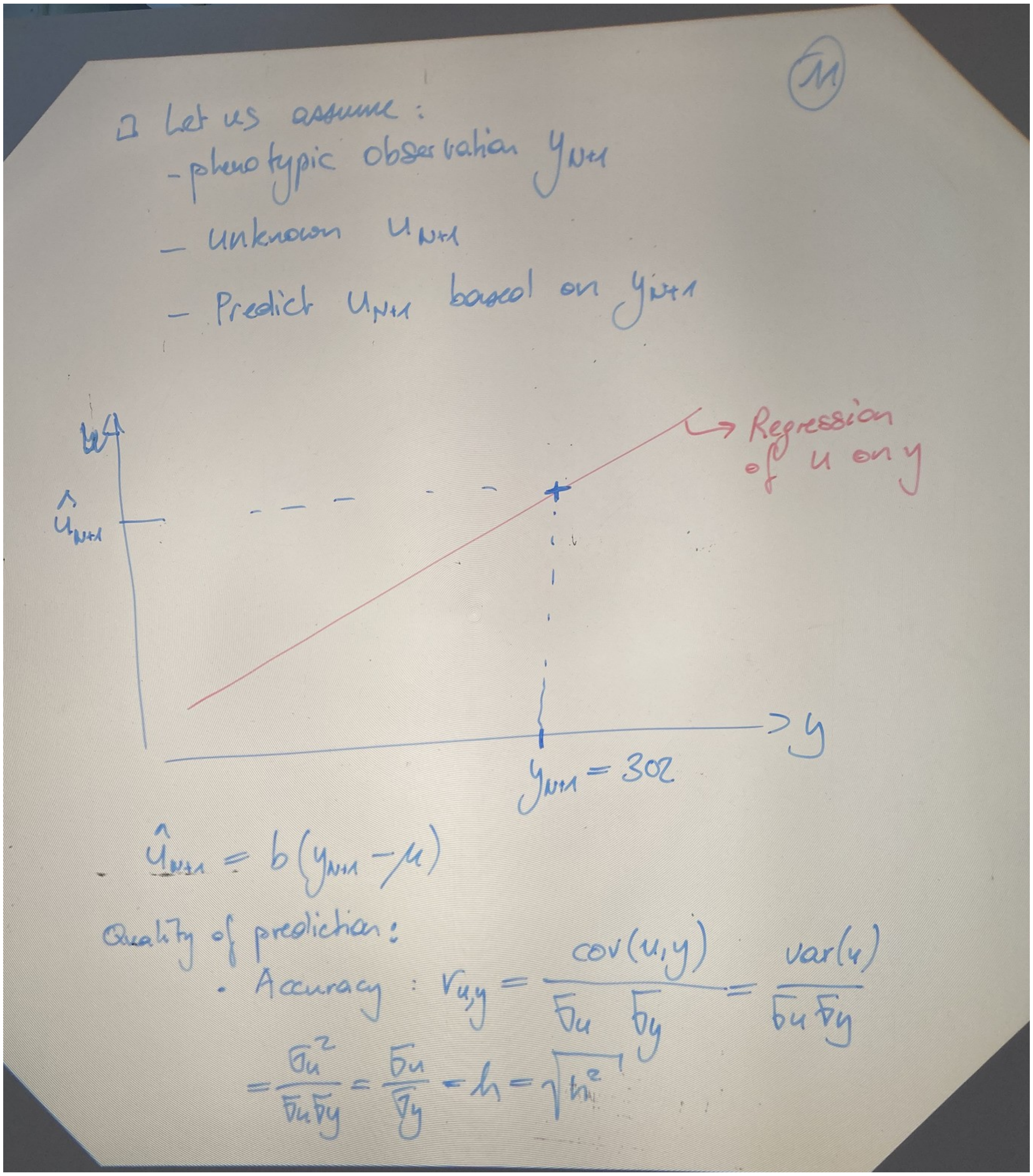
OHP Picture 9



OHP Picture 10



OHP Picture 11



OHP Picture 12

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□ Variability of predictions:

$$\mathrm{var}(\hat{u}_i) = \mathrm{var}(b(y_i - \overset{\text{const}}{\mu}))$$

$$= \mathrm{var}(by_i - b\mu)$$

$$= \mathrm{var}(by_i) + \underbrace{\mathrm{var}(b\mu)}_{=0} - 2\underbrace{\mathrm{cov}(by_i, b\mu)}_{=0}$$

$$= \mathrm{var}(by_i)$$

$$= b^2 \cdot \mathrm{var}(y_i) = h^4 \sigma_y^2$$

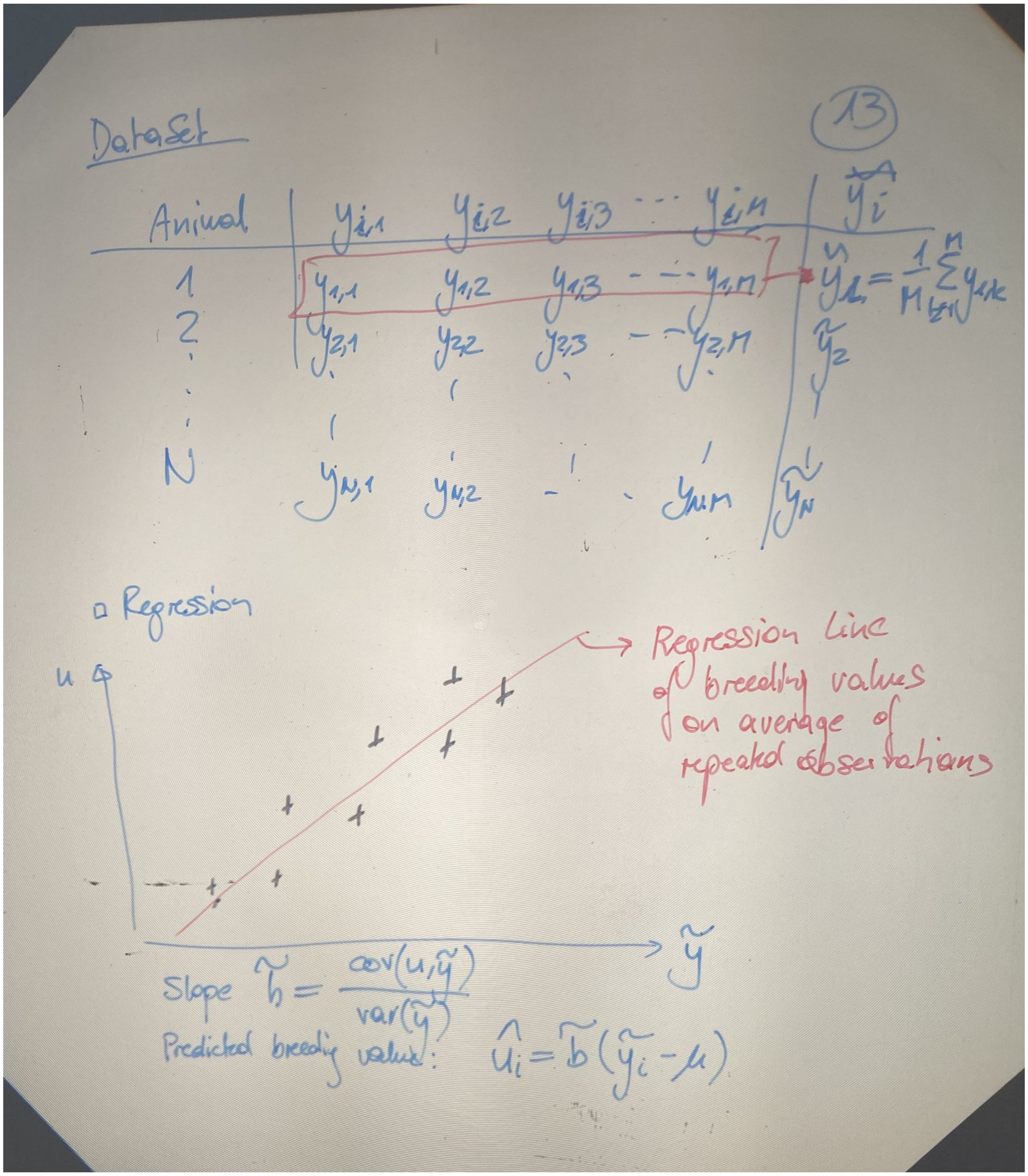
$$= \frac{\sigma_u^4}{\sigma_p^4} \cdot \sigma_y^2 = \frac{\sigma_u^4}{\sigma_u^2} = h^2 \sigma_u^2$$

$\mathrm{var}(\hat{u}_i)$ should approximate $\mathrm{var}(u) = \sigma_u^2$

Extension:

- More observations per animal
  $\Rightarrow$ Repeated measure

OHP Picture 13



OHP Picture 14

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- With repeated measurements
  - Separation of environment into permanent environment (pe) and into non-permanent environment (te)

$$y_{ik} = \mu + u_i + e_{ik}$$

$e_{ik} \rightarrow pe_i$: constant over all measurements for animal i, e.g. the same herd, ...

$e_{ik} \rightarrow te_{ik}$: change with every measurement.

$$\tilde{y}_i = \frac{1}{M}\sum_{k=1}^{M} y_{ik} = \frac{1}{M}\sum_{k=1}^{M}\left[\mu + u_i + pe_i + te_{ik}\right]$$

$$= \frac{1}{M}\left\{\sum_{k=1}^{M}\mu + \sum_{k=1}^{M}u_i + \sum_{k=1}^{M}pe_i + \sum_{k=1}^{M}te_{ik}\right\}$$

$$= \mu + u_i + pe_i + \frac{1}{M}\sum_{k=1}^{M}te_{ik}$$

OHP Picture 15

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$$\tilde{b} = \frac{cov(u,\tilde{y})}{var(\tilde{y})}$$

$$cov(u_i, \tilde{y}_i) = cov\left(u_i, \mu + u_i + pe_i + \frac{1}{M}\sum_{k=1}^{M} te_{ik}\right)$$

$$= \underbrace{cov(u_i,\mu)}_{=0} + cov(u_i,u_i) + \underbrace{cov(u_i,pe_i)}_{=0}$$

$$+ \underbrace{cov\left(u_i, \frac{1}{M}\sum_{k=1}^{M} te_{ik}\right)}_{=0}$$

$$= cov(u_i,u_i) = \sigma_u^2$$

$$var(\tilde{y}) = var\left(\underbrace{\mu}_{=0} + u_i + pe_i + \frac{1}{M}\sum_{k=1}^{M} te_{ik}\right)$$

$$= var(\mu) + var(u_i) + var(pe_i) + var\left(\frac{1}{M}\sum_{k=1}^{M} te_{ik}\right)$$

covariances are $= 0$

$$+ 2\,cov(\mu, u_i) + \ldots$$

$$= var(u_i) + var(pe_i) + \overbrace{var\left(\frac{1}{M}\sum_{k=1}^{M} te_{ik}\right)}^{1}$$

$$= var(u_i) + var(pe_i) + \frac{1}{M} var(te_i)$$

$$= 2$$

OHP Picture 16

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$$\mathrm{var}\left(\frac{1}{M}\sum_{k=1}^{M} te_{ik}\right) = \frac{1}{M^2}\,\mathrm{var}\left(\sum_{k=1}^{M} te_{ik}\right)$$

$$= \frac{1}{M^2}\,\mathrm{var}\left(te_{i1} + te_{i2} + \dots + te_{iM}\right)$$

$$= \frac{1}{M^2}\left\{\overbrace{\mathrm{var}(te_{i1})}^{=\mathrm{var}(te_i)} + \mathrm{var}(te_{i2}) + \dots + \mathrm{var}(te_{iM}) \underbrace{+\, 2\,\mathrm{cov}(\dots)}_{=0}\right\}$$

$$= \frac{1}{M^2}\cdot M \cdot \mathrm{var}(te_i) = \frac{1}{M}\,\mathrm{var}(te_i)$$

$$y_{ik} = \mu + u_i + pe_i + te_{ik}$$

$$\mathrm{var}(y_{ik}) = \mathrm{var}(\mu + u_i + pe_i + te_{ik})$$

$$= \underbrace{\mathrm{var}(u_i) + \mathrm{var}(pe_i)}_{\text{permanent}} + \mathrm{var}(te_{ik}) = \sigma_y^2$$

Repeatability $t = \dfrac{\mathrm{var}(u_i) + \mathrm{var}(pe_i)}{\sigma_y^2}$

OHP Picture 17

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$$\mathrm{var}(\tilde{y}) = \underline{\mathrm{var}(u_i) + \mathrm{var}(pe_i)} + \frac{1}{M}\underline{\mathrm{var}(te_i)}$$

$$\text{with} \quad t = \frac{\mathrm{var}(u_i) + \mathrm{var}(pe_i)}{\sigma_y^2} \Rightarrow \mathrm{var}(u_i) + \mathrm{var}(pe_i) = t \cdot \sigma_y^2$$

$$1 - t = \frac{\mathrm{var}(te_i)}{\sigma_y^2} \Rightarrow \mathrm{var}(te_i) = (1-t)\sigma_y^2$$

$$\Rightarrow \mathrm{var}(\tilde{y}) = \underline{t \cdot \sigma_y^2} + \frac{1}{M}\underline{(1-t)\sigma_y^2}$$

$$= \ldots$$

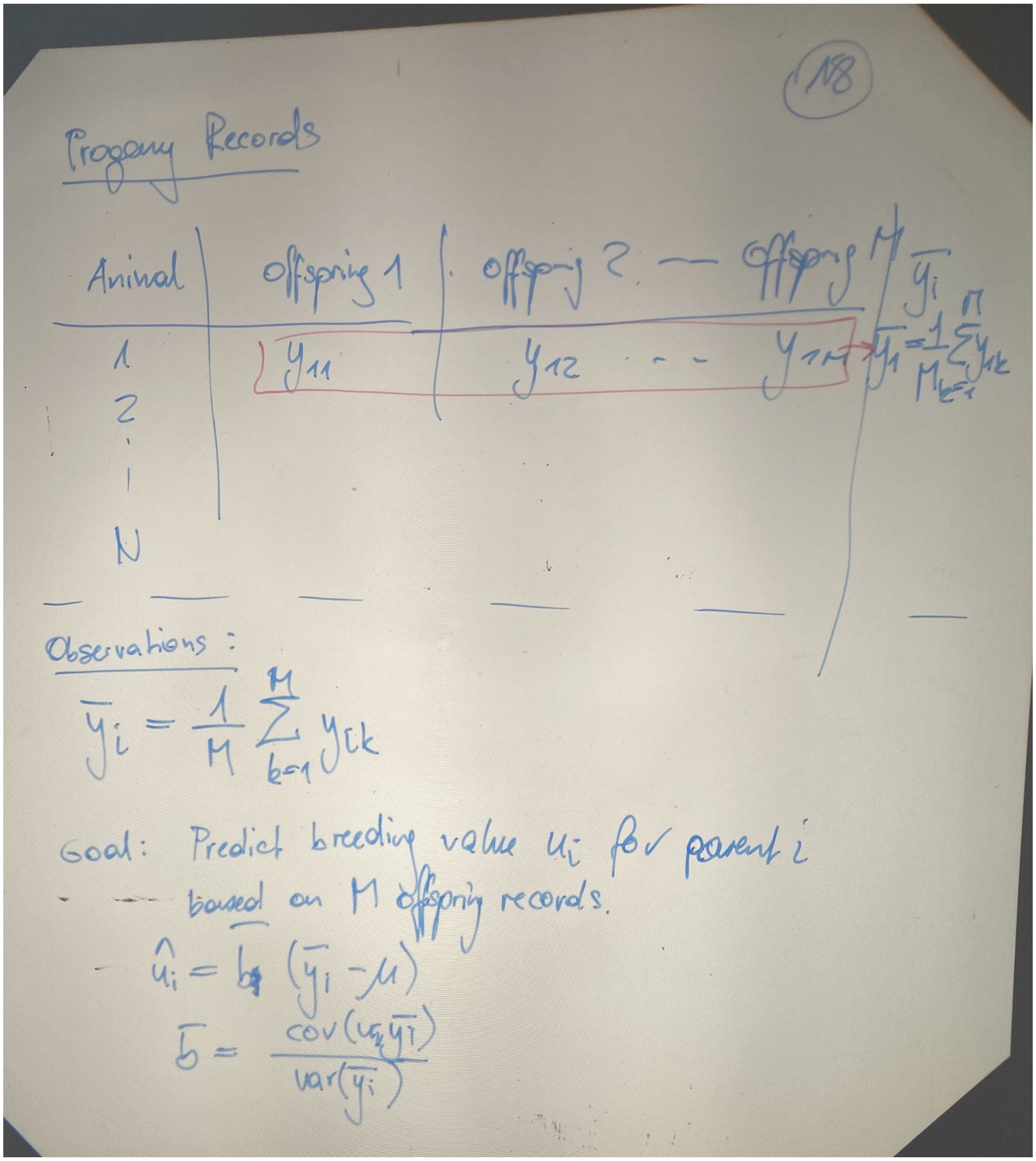
$$= \frac{1 + (M-1)t}{M}\sigma_y^2$$

$$\tilde{b} = \frac{\mathrm{cov}(u, \tilde{y})}{\mathrm{var}(\tilde{y})} = \frac{\sigma_u^2}{\frac{1+(M-1)t}{M}\sigma_y^2} = \frac{M\sigma_u^2}{[1 + (M-1)t]\sigma_y^2}$$

$$= \frac{M\sigma_u^2}{(1 + Mt - t)\sigma_y^2} = \frac{Mh^2}{1 + (M-1)t}$$

$$a_i = \tilde{b}(\tilde{y}_i - \mu) = \frac{Mh^2}{1+(M-1)t}(\tilde{y}_i - \mu)$$

OHP Picture 18



OHP Picture 19

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$$\bar{b} = \frac{cov(u_i, \bar{y}_i)}{var(\bar{y}_i)} \; ; \; \bar{y}_i = \frac{1}{M}\sum_{k=1}^{M} y_{ik}$$

$$= \frac{1}{M}\sum_{k=1}^{M} \mu + u_k + e_{ik}$$

$$= \frac{1}{M}\sum_{k=1}^{M} \left\{ \mu + \frac{1}{2} u_i + \frac{1}{2} u_{dk} + m_k + e_{ik} \right\}$$

$$cov(u_i, \bar{y}_i) =$$

$$= \mu + \underbrace{\frac{1}{2} u_i}_{\text{permanent}} + \frac{1}{2}\frac{1}{M}\sum_{k=1}^{M} u_{dk} + \frac{1}{2}\frac{1}{M}\sum_{k=1}^{M} m_k + \frac{1}{M}\sum_{k=1}^{M} e_{ik}$$

$$cov(u_i, \bar{y}_i) = cov(u_i, \tfrac{1}{2} u_i) = \tfrac{1}{2} cov(u_i, u_i) = \tfrac{1}{2}\sigma_u^2$$

$$var(\bar{y}_i) = t \cdot \sigma_y^2 + \frac{1}{M}(1-t)\sigma_y^2$$

$$\text{with } t = \frac{var(\frac{1}{2} u_i)}{\sigma_y^2} = \frac{\frac{1}{4}\sigma_u^2}{\sigma_y^2} = \tfrac{1}{4} h^2$$

$$var(\bar{y}_i) = \left[\frac{h^2}{4} + \frac{1}{M}\left(1 - \frac{h^2}{4}\right)\right]\sigma_y^2$$

$$\bar{b} = \frac{\frac{1}{2}\sigma_u^2}{\left[\frac{h^2}{4} + \frac{1}{M}\frac{3h^2}{4}\right]\sigma_y^2} = \ldots = \frac{2n}{n+k}$$

OHP Picture 20

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$$\hat{u}_i = \bar{b}\,(\bar{y}_i - \mu)$$

$$= \frac{2n}{n+k}\,(\bar{y}_i - \mu) \quad \text{with } k = \frac{4-h^2}{h^2}$$