

OHP Picture 1

Recap: 2023-10-20

□ New Model: (Infinitesimal model) ①

$y_{ij} = \mu_i + u_i + e_{ij}$

- μ_i : Known environment (herd, season, age, ...)
- u_i : breeding value of animal i
- e_{ij} : unknown part of environment, plus dominance epistasis

(Later: Genomic selection: polygenic model
large but finite number of loci)

□ Decomposition of u_i into parent average plus mendelian sampling term (m_i)

$u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + m_i$

- u_i : breeding value of animal i
- u_s : breeding value of s
- u_d : breeding value of d
- m_i : breeding value of animal i

Animals s and d are parents of i

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□ Decomposition :

$$u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + m_i$$

- m_i is a deviation, this means over a large number (N) of offspring from parents s and d , the average over all mendelian sampling terms (m_i) is zero.

$$\Rightarrow \frac{1}{N} \sum_{i=1}^N m_i = \frac{1}{N} (m_1 + m_2 + \dots + m_N) = 0$$

$$u_1 = \frac{1}{2} u_s + \frac{1}{2} u_d + m_1$$

$$u_2 = \frac{1}{2} u_s + \frac{1}{2} u_d + m_2$$

$$u_N = \frac{1}{2} u_s + \frac{1}{2} u_d + m_N$$

$$\frac{1}{N} \sum_{i=1}^N m_i = 0$$

$$\frac{1}{N} \sum_{i=1}^N u_i = \frac{1}{N} (u_1 + u_2 + \dots + u_N)$$

$$\frac{1}{2} u_s + \frac{1}{2} u_d$$

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□ Verify

- Single locus G
- Different cases according to parent genotypes

□ Case 1: Parents s and d are homozygous

s d

i

Ditto for G2

$$\left. \begin{aligned} u_s &= 2q\alpha \\ u_d &= 2q\alpha \\ u_i &= 2q\alpha \end{aligned} \right\}$$

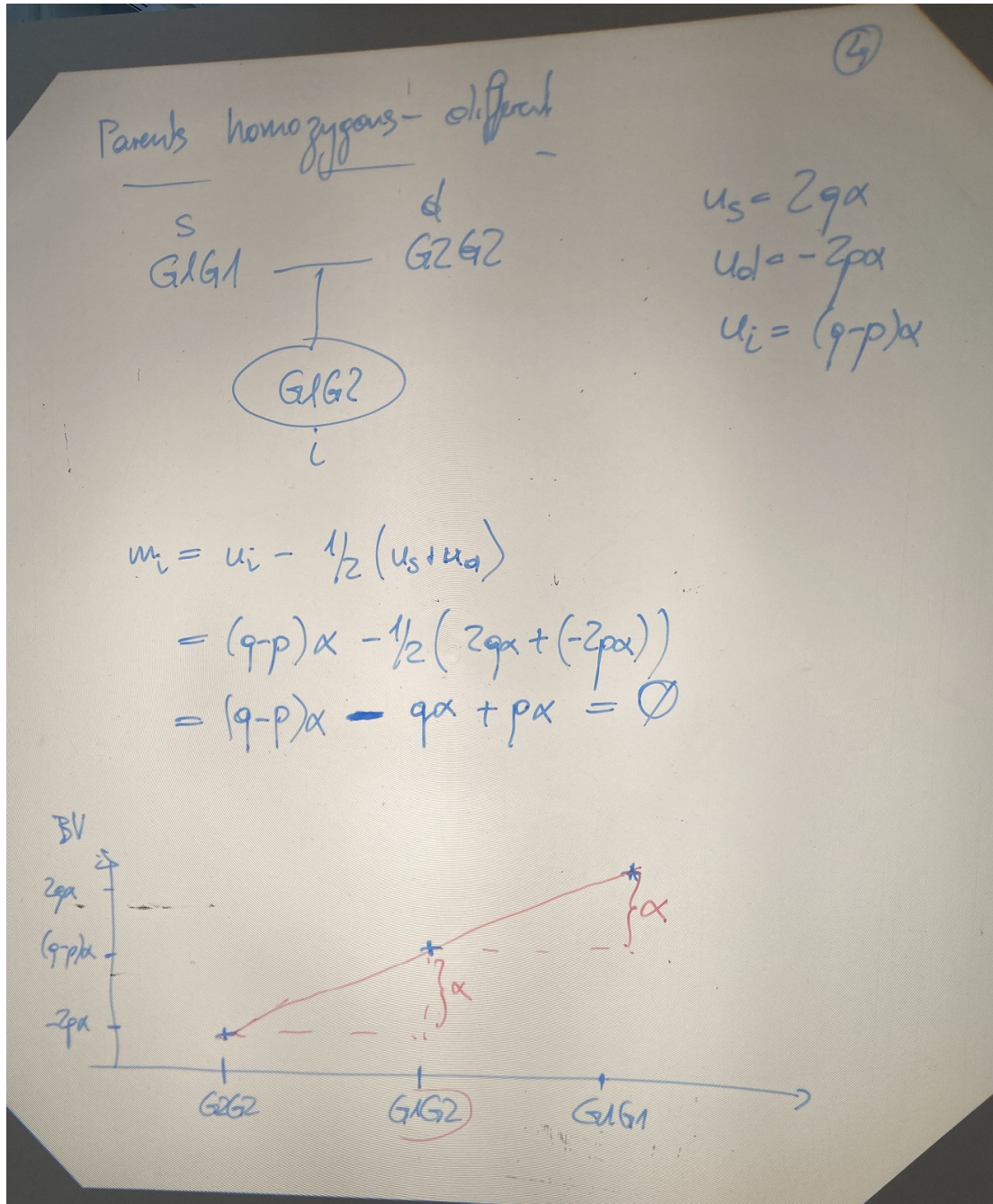
$$u_i = \frac{1}{2}u_s + \frac{1}{2}u_d + w_i$$

$$w_i = 2q\alpha - \left(\frac{1}{2}2q\alpha + \frac{1}{2}2q\alpha\right)$$

$$= 2q\alpha - (q\alpha + q\alpha) = 0$$

$$\frac{1}{N} \sum_{i=1}^N w_i = \frac{1}{N} (w_1 + w_2 + \dots + w_N) = \frac{1}{N} (0 + 0 + \dots + 0) = 0$$

OHP Picture 4



OHP Picture 5

⑤

o Mixed: one parent homozygous
other parent heterozygous

1000 G1G1
⇒ 500 G1G1
500 G1G2

$$u_s = 2q\alpha$$

$$u_d = (q+p)\alpha$$

$$\frac{1}{2}(u_s + u_d) =$$

$$= \frac{1}{2}(2q\alpha + (q+p)\alpha)$$

$$= q\alpha + \frac{1}{2}q\alpha - \frac{1}{2}p\alpha$$

$$= \frac{3}{2}q\alpha - \frac{1}{2}p\alpha$$

$$u_i = \frac{1}{2}u_s + \frac{1}{2}u_d + m_i$$

$$m_i = u_i - \frac{1}{2}(u_s + u_d)$$

$$= 2q\alpha - \left(\frac{3}{2}q\alpha - \frac{1}{2}p\alpha\right) = 2q\alpha - \frac{3}{2}q\alpha + \frac{1}{2}p\alpha$$

$$= \frac{1}{2}q\alpha + \frac{1}{2}p\alpha$$

$$= \frac{1}{2}\alpha$$

$$u_j = u_j - \frac{1}{2}(u_s + u_d)$$

$$= (q-p)\alpha - \left(\frac{3}{2}q\alpha - \frac{1}{2}p\alpha\right)$$

$$= q\alpha - p\alpha - \frac{3}{2}q\alpha + \frac{1}{2}p\alpha = -\frac{1}{2}q\alpha - \frac{1}{2}p\alpha$$

$$= -\frac{1}{2}\alpha$$

OHP Picture 6

$w_1 = \frac{1}{2}\alpha$
 $w_2 = -\frac{1}{2}\alpha$ } $N = 1000$ offspring (6)

$\frac{1}{N} \sum_{k=1}^N w_k = \frac{1}{2} \cdot \frac{1}{2}\alpha + \frac{1}{2}(-\frac{1}{2}\alpha) = 0$

Other Mixed parent

S	d
G2G2	G1G2

Heterozygous

S	d
G1G2	G1G2

Parent-average of breeding values: $\frac{1}{2}(u_S + u_d)$

Exercise 06

OHP Picture 7

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□ Breeding values for complete populations
parents: s, d
offspring: i, j

$cov(u_s, u_i)$

use decomposition:

$$u_i = \frac{1}{2}u_s + \frac{1}{2}u_d + m_i$$

$cov(u_s, u_i) = cov(u_s, (\frac{1}{2}u_s + \frac{1}{2}u_d + m_i))$
 $= cov(u_s, \frac{1}{2}u_s) + cov(u_s, \frac{1}{2}u_d)$
 $+ cov(u_s, m_i)$

$cov(u_i, u_j)$

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graph TD; s((s)) --- d((d)); s --- j((j)); d --- i((i)); i --- j;
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□ Infinitesimal Model: For animal i

$$y_{ij} = \mu_i + u_i + e_{ij}^*$$

i u_i (and e_{ij}^*)
unknown,
Goal: prediction for u_i
based on y_{ij}

□ Data set:

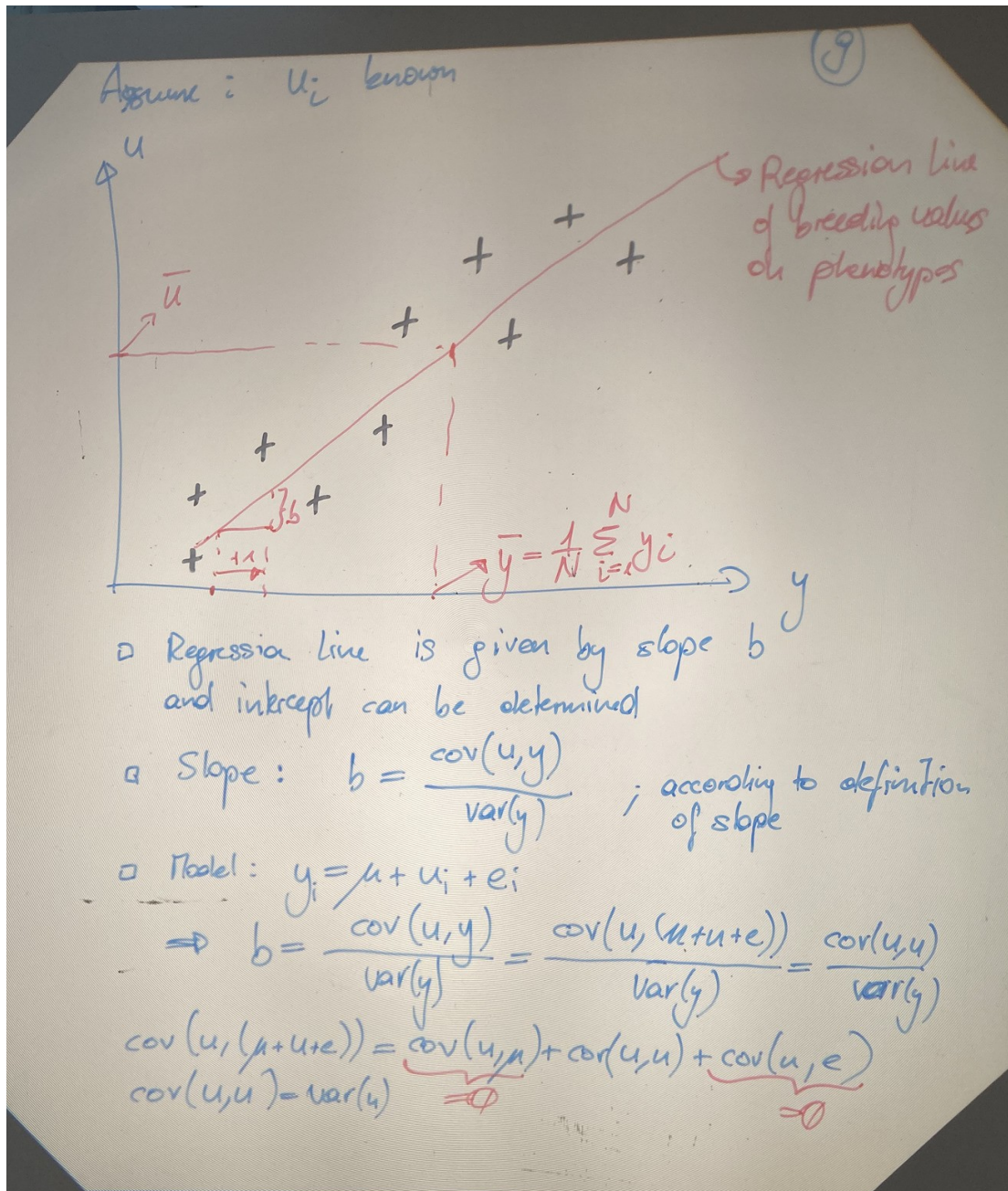
Animal	y (weight in kg)
1	$y_1 = 250$
2	$y_2 = 310$
⋮	⋮
⋮	⋮
N	$y_N = 293$

} One observation per animal
weight after 1 year

□ Assume:

- Use selection of parents such that weight after 1 year is improved.
- Predict breeding values based on available information
Call predicted breeding value for animal i : \hat{u}_i

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$$b = \frac{\text{cov}(u, u)}{\text{var}(y)} = \frac{\text{var}(u)}{\text{var}(y)} = \frac{\sigma_u^2}{\sigma_y^2} = h^2$$

with h^2 : heritability $h^2 < 1$

Intercept: $y_i = \mu + u_i + e_i$

$$\text{var}(y) = \text{var}(\mu) + \text{var}(u) + \text{var}(e) + 2\text{Cov}(\dots)$$

$$= \text{var}(u) + \text{var}(e)$$

Regression: ~~$u_i = \frac{1}{2}(y_i - \mu)$~~

allowing for slope b :

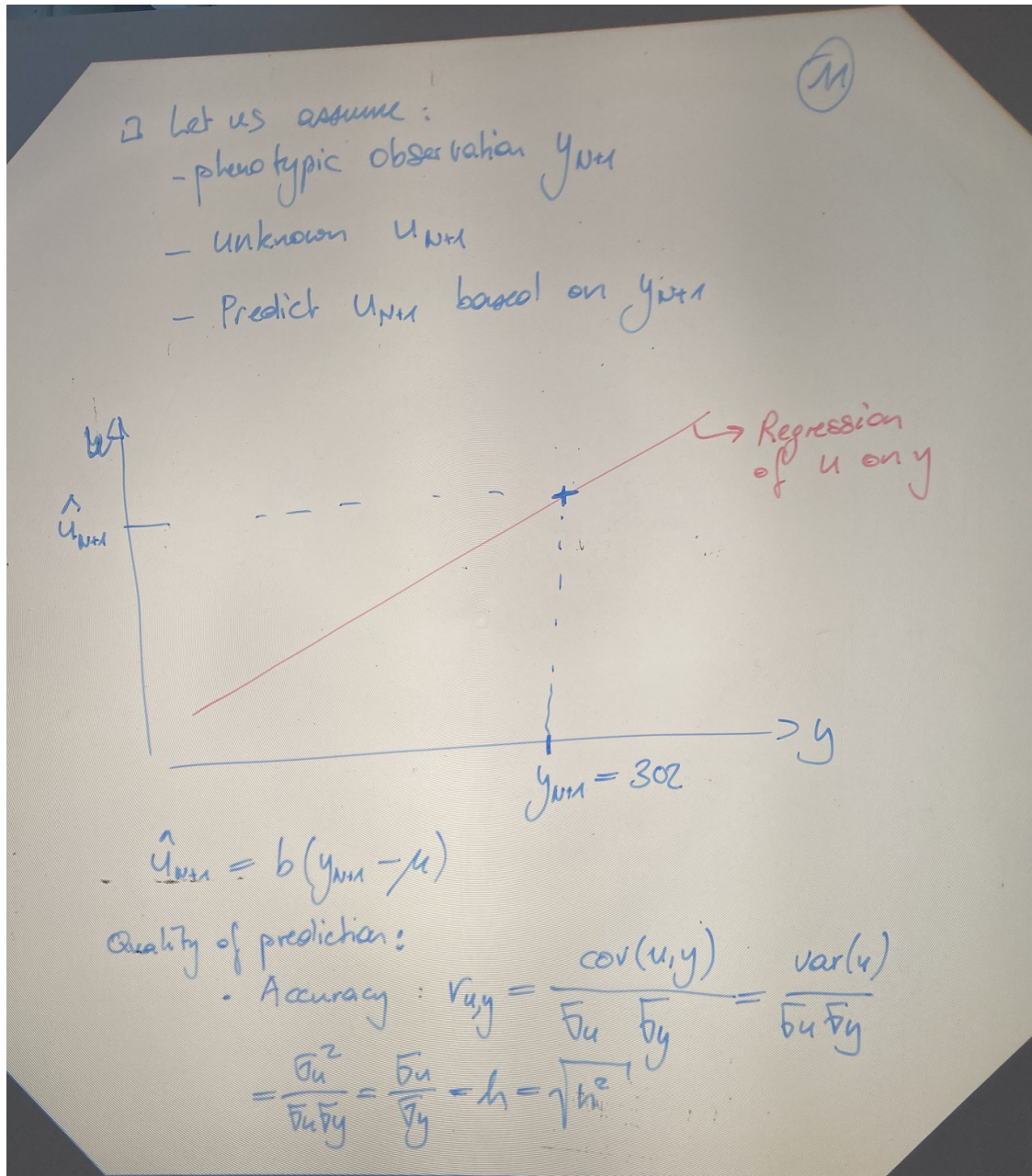
Regression: $y = b(x - \bar{x}_c)$

$u = b(y - \mu)$

$\bar{u} = b(\bar{y} - \mu)$

average = 0 $\Rightarrow \mu = \bar{y} = \frac{1}{N} \sum_{i=1}^N y_i$

OHP Picture 11



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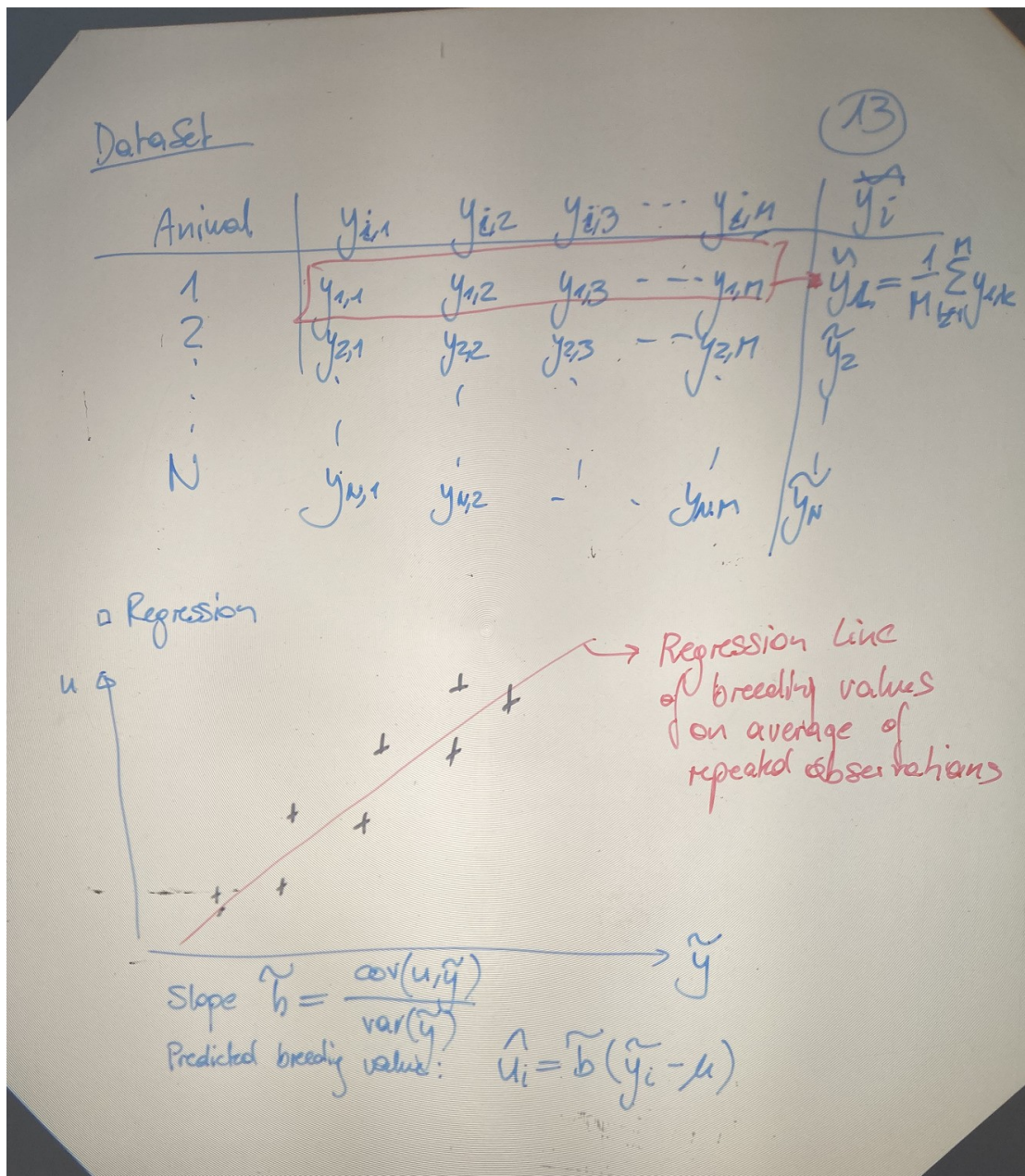
□ Variability of predictions:

$$\begin{aligned} \text{var}(\hat{u}_i) &= \text{var}(b(y_i - \mu)) \\ &= \text{var}(by_i - b\mu) \\ &= \text{var}(by_i) + \underbrace{\text{var}(b\mu)}_{=0} - 2 \underbrace{\text{cov}(by_i, b\mu)}_{=0} \\ &= \text{var}(by_i) \\ &= b^2 \cdot \text{var}(y_i) = h^4 \sigma_y^2 \\ &= \frac{\sigma_u^4}{\sigma_p^4} \cdot \sigma_y^2 = \frac{\sigma_u^4}{\sigma_u^2} = h^2 \sigma_u^2 \end{aligned}$$

$\text{var}(\hat{u}_i)$ should approximate $\text{var}(u) = \sigma_u^2$

Extension:

- More observations per animal
 \Rightarrow Repeated measure



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□ With repeated measurements

- Separation of environment into permanent environment (pe) and into non-permanent environment (te)

$$y_{ik} = \mu + u_i + e_{ik}$$

\swarrow \swarrow
 pe_i te_{ik}

constant over all measurements for animal i
 e.g. the same herd

\rightarrow change with every measurement.

$$\begin{aligned} \bar{y}_i &= \frac{1}{M} \sum_{k=1}^M y_{ik} = \frac{1}{M} \sum_{k=1}^M [\mu + u_i + pe_i + te_{ik}] \\ &= \frac{1}{M} \left\{ \sum_{k=1}^M \mu + \sum_{k=1}^M u_i + \sum_{k=1}^M pe_i + \sum_{k=1}^M te_{ik} \right\} \\ &= \mu + u_i + pe_i + \frac{1}{M} \sum_{k=1}^M te_{ik} \end{aligned}$$

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$$\tilde{b} = \frac{\text{cov}(u_i, \tilde{y}_i)}{\text{var}(\tilde{y}_i)}$$

$$\begin{aligned} \text{cov}(u_i, \tilde{y}_i) &= \text{cov}\left(u_i, \mu + u_i + pe_i + \frac{1}{M} \sum_{k=1}^M te_{ik}\right) \\ &= \text{cov}(u_i, \mu) + \text{cov}(u_i, u_i) + \text{cov}(u_i, pe_i) \\ &\quad + \text{cov}\left(u_i, \frac{1}{M} \sum_{k=1}^M te_{ik}\right) \\ &= \text{cov}(u_i, u_i) = \sigma_u^2 \end{aligned}$$

$$\begin{aligned} \text{var}(\tilde{y}_i) &= \text{var}\left(\mu + u_i + pe_i + \frac{1}{M} \sum_{k=1}^M te_{ik}\right) \\ &= \text{var}(\mu) + \text{var}(u_i) + \text{var}(pe_i) + \text{var}\left(\frac{1}{M} \sum_{k=1}^M te_{ik}\right) \\ &\quad + 2 \text{cov}(\mu, u_i) + \dots \\ &= \text{var}(u_i) + \text{var}(pe_i) + \text{var}\left(\frac{1}{M} \sum_{k=1}^M te_{ik}\right) \\ &= \text{var}(u_i) + \text{var}(pe_i) + \frac{1}{M} \text{var}(te_1) \\ &= \textcircled{2} \end{aligned}$$

covariances are = 0

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OHP Picture 16

$$\text{var}\left(\frac{1}{M} \sum_{k=1}^M t_{ik}\right) = \frac{1}{M^2} \text{var}\left(\sum_{k=1}^M t_{ik}\right) \quad (16)$$

$$= \frac{1}{M^2} \text{var}(t_{i1} + t_{i2} + \dots + t_{iM})$$

$$= \frac{1}{M^2} \left\{ \underbrace{\text{var}(t_{i1}) + \text{var}(t_{i2}) + \dots + \text{var}(t_{iM})}_{= \text{var}(t_{ci})} + \underbrace{2 \text{Cov}(\dots)}_{= 0} \right\}$$

$$= \frac{1}{M^2} \cdot M \cdot \text{var}(t_{ci}) = \frac{1}{M} \text{var}(t_{ci})$$

$$y_{ik} = \mu + u_i + p_{ei} + t_{ik}$$

$$\begin{aligned} \text{var}(y_{ik}) &= \text{var}(\mu + u_i + p_{ei} + t_{ik}) \\ &= \underbrace{\text{var}(u_i) + \text{var}(p_{ei})}_{\text{permanent}} + \text{var}(t_{ik}) = \sigma_y^2 \end{aligned}$$

$$\text{Repeatability } r = \frac{\text{var}(u_i) + \text{var}(p_{ei})}{\sigma_y^2}$$

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$$\text{var}(\tilde{y}) = \frac{\text{var}(u_i) + \text{var}(pe_i)}{n} + \frac{1}{n} \text{var}(k_i) \quad (17)$$

with $t = \frac{\text{var}(u_i) + \text{var}(pe_i)}{\sigma_y^2} \Rightarrow \text{var}(u_i) + \text{var}(pe_i) = t \cdot \sigma_y^2$

$$1-t = \frac{\text{var}(k_i)}{\sigma_y^2} \Rightarrow \text{var}(k_i) = (1-t) \sigma_y^2$$

$$\Rightarrow \text{var}(\tilde{y}) = \frac{t \cdot \sigma_y^2}{n} + \frac{1}{n} (1-t) \sigma_y^2$$

$$= \frac{1 + (n-1)t}{n} \sigma_y^2$$

$$\tilde{b} = \frac{\text{cov}(u, \tilde{y})}{\text{var}(\tilde{y})} = \frac{\sigma_u^2}{\frac{1 + (n-1)t}{n} \sigma_y^2} = \frac{n \sigma_u^2}{[1 + (n-1)t] \sigma_y^2}$$

$$= \frac{n \sigma_u^2}{(1 + n t - t) \sigma_y^2} = \frac{n h^2}{1 + (n-1)t}$$

$$q_i - \tilde{b} (\tilde{y}_i - \mu) = \frac{n h^2}{1 + (n-1)t} (\tilde{y}_i - \mu)$$

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Progeny Records

Animal	offspring 1	offspring 2	...	offspring M	\bar{y}_i
1	y_{11}	y_{12}	...	y_{1M}	$\bar{y}_i = \frac{1}{M} \sum_{k=1}^M y_{ik}$
2					
i					
N					

Observations:

$$\bar{y}_i = \frac{1}{M} \sum_{k=1}^M y_{ik}$$

Goal: Predict breeding value u_i for parent i based on M offspring records.

$$\hat{u}_i = b (\bar{y}_i - \mu)$$

$$b = \frac{\text{cov}(u_i, \bar{y}_i)}{\text{var}(\bar{y}_i)}$$

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$$\bar{b} = \frac{\text{cov}(u_i, \bar{y}_i)}{\text{var}(\bar{y}_i)} ; \bar{y}_i = \frac{1}{M} \sum_{k=1}^M y_{ik}$$

$$\text{cov}(u_i, \bar{y}_i) = \frac{1}{M} \sum_{k=1}^M (\mu + u_k + e_{ik})$$

$$= \frac{1}{M} \sum_{k=1}^M \left\{ \mu + \frac{1}{2} u_i + \frac{1}{2} u_k + \mu_k + e_{ik} \right\}$$

$$= \underbrace{\mu + \frac{1}{2} u_i}_{\text{permanent}} + \frac{1}{2} \frac{1}{M} \sum_{k=1}^M u_k + \frac{1}{2} \frac{1}{M} \sum_{k=1}^M \mu_k + \frac{1}{2} \frac{1}{M} \sum_{k=1}^M e_{ik}$$

$$\rightarrow \text{cov}(u_i, \bar{y}_i) = \text{cov}(u_i, \frac{1}{2} u_i) = \frac{1}{2} \text{cov}(u_i, u_i)$$

$$\text{var}(\bar{y}_i) = t \cdot \bar{\sigma}_y^2 + \frac{1}{M} (1-t) \bar{\sigma}_y^2 = \frac{1}{2} \bar{\sigma}_y^2$$

with $t = \frac{\text{var}(\frac{1}{2} u_i)}{\bar{\sigma}_y^2} = \frac{1/4 \bar{\sigma}_y^2}{\bar{\sigma}_y^2} = 1/4$

$$\text{var}(\bar{y}_i) = \left[\frac{1}{4} + \frac{1}{M} \left(1 - \frac{1}{4} \right) \right] \bar{\sigma}_y^2$$

$$\bar{b} = \frac{1/2 \bar{\sigma}_y^2}{\left[\frac{1}{4} + \frac{1}{M} \frac{3/4}{4} \right] \bar{\sigma}_y^2} = \dots = \frac{2n}{n+k}$$

OHP Picture 20

$$\begin{aligned}\hat{u}_i &= \bar{b} (\bar{y}_i - \mu) \\ &= \frac{2n}{n+k} (\bar{y}_i - \mu) \quad \text{with } b = \frac{4-h^2}{h^2}\end{aligned}$$

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