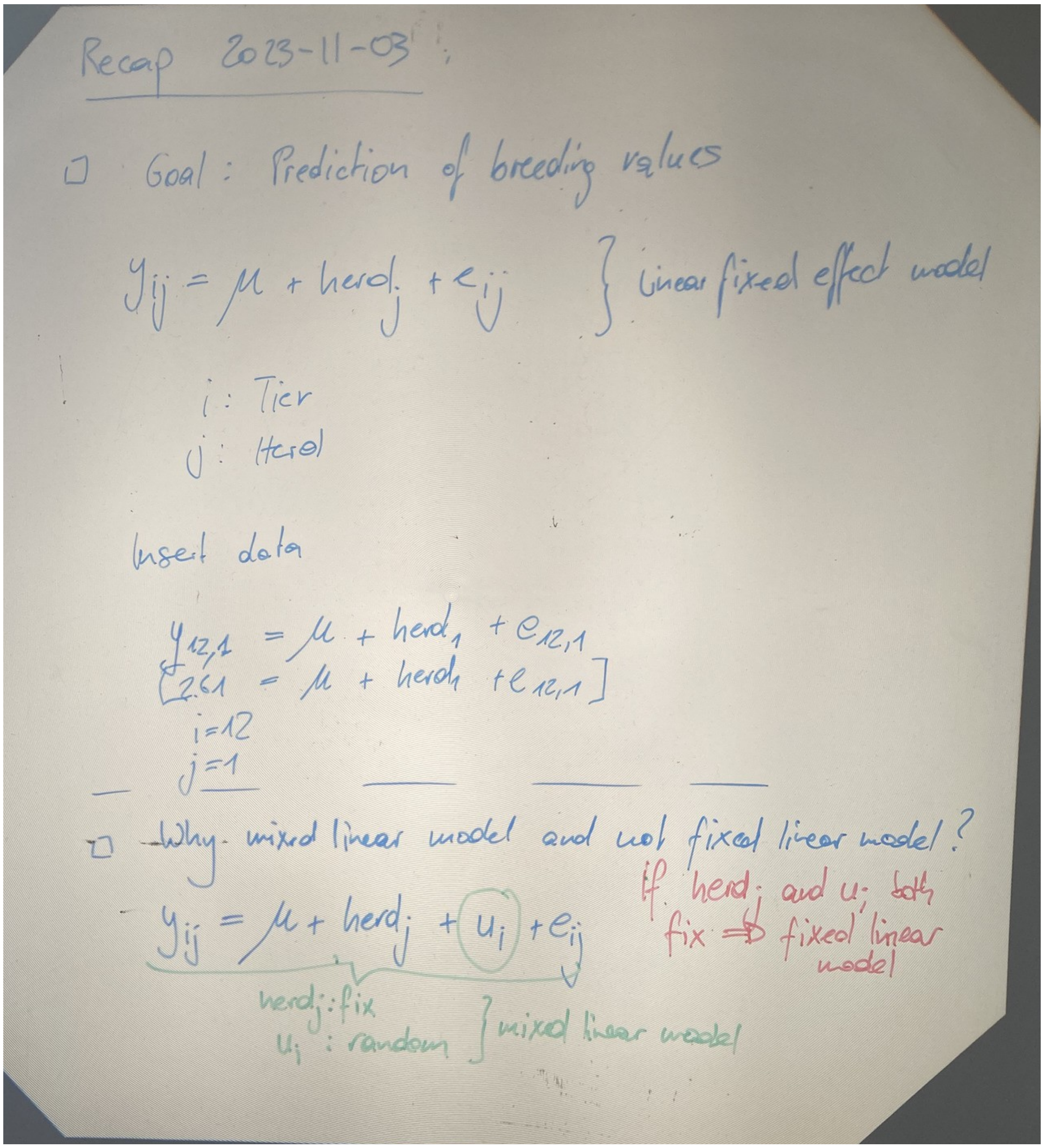
OHP Picture 1



Recap 2023-11-03:

□ Goal: Prediction of breeding values

$y_{ij} = \mu + herd_j + e_{ij}$ } linear fixed effect model

i: Tier

j: Herd

Insert data

$y_{12,1} = \mu + herd_1 + e_{12,1}$

$[2.61 = \mu + herd_1 + e_{12,1}]$

$i = 12$

$j = 1$

□ Why mixed linear model and not fixed linear model?

$y_{ij} = \mu + herd_j + u_i + e_{ij}$

if $herd_j$ and u_i both fix ⇒ fixed linear model

$herd_j$: fix

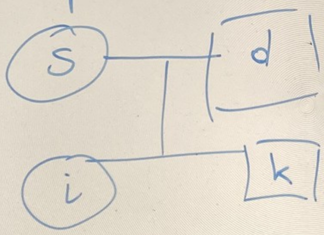
u_i: random } mixed linear model

OHP Picture 2

□ Breeding Values

$$y_{ij} = \mu + herd_j + u_i + e_{ij}$$

- Observation: weight of animal i in herd j
- u_i must be random, because $cov(u_i, u_k) \neq 0$ if animals i and k are related
- Example: animals i and k are full-sibs:



- For animal k: $y_{kl} = \mu + herd_l + u_j + e_{kl}$

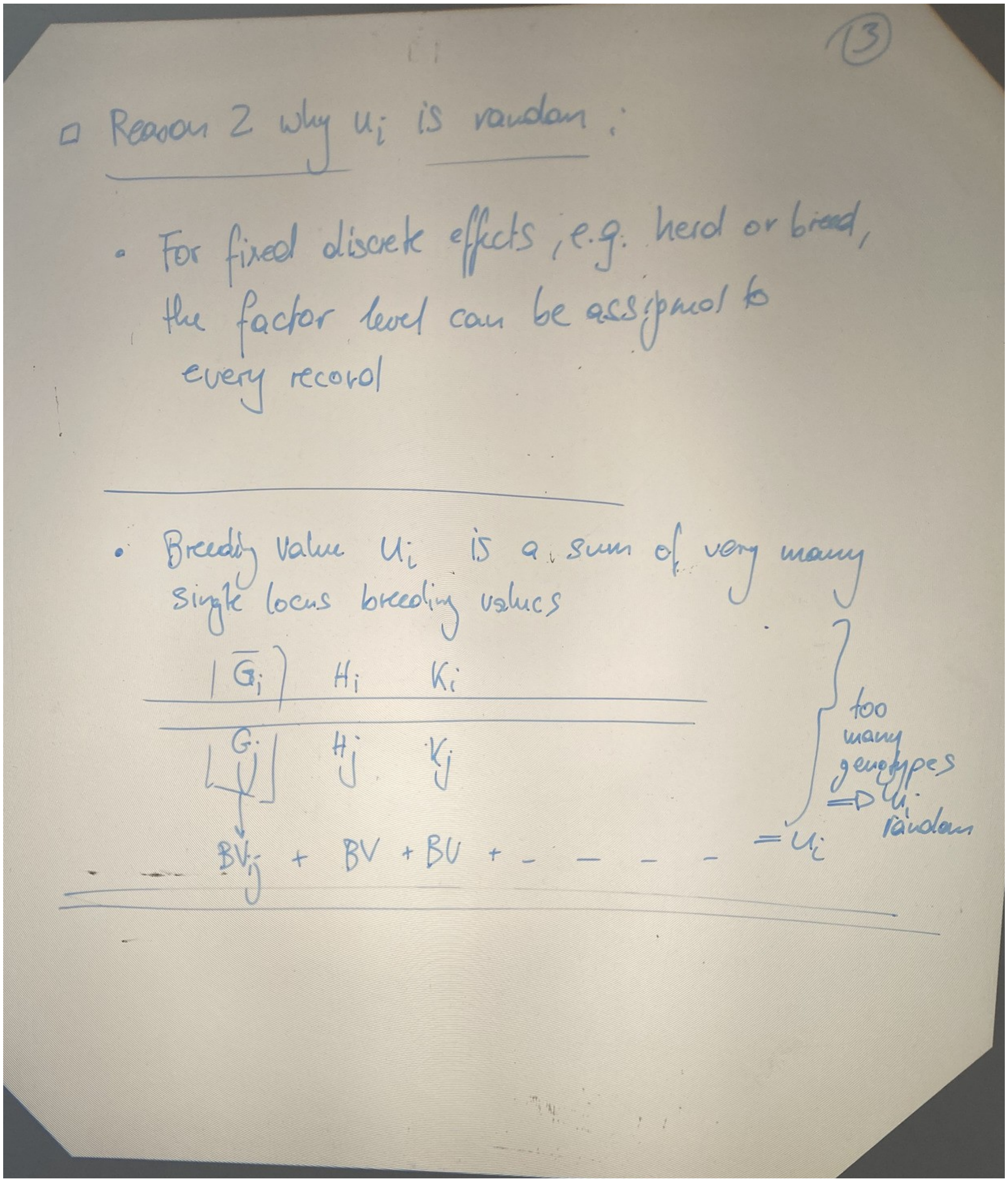
$cov(u_i, u_k) \neq 0$, because of decomposition of breeding values

$$\left.\begin{aligned} u_i &= \tfrac{1}{2} u_s + \tfrac{1}{2} u_d + m_i \\ u_k &= \tfrac{1}{2} u_s + \tfrac{1}{2} u_d + m_k \end{aligned}\right\} cov(u_i, u_k) \neq 0$$

$$\begin{aligned} cov(u_i, u_k) &= cov\left(\left[\tfrac{1}{2} u_s + \tfrac{1}{2} u_d + m_i\right], \left[\tfrac{1}{2} u_s + \tfrac{1}{2} u_d + m_k\right]\right) \\ &= \tfrac{1}{4} var(u_s) + \tfrac{1}{4} var(u_d) = \tfrac{1}{2} \sigma_u^2 \end{aligned}$$

provided s and d not related.

OHP Picture 3



OHP Picture 4

(4)

- Regression

$$y_i = \mu + b \cdot \text{breast\_circumference}_i + e_i$$

herd u_i

- Mixed-linear Model

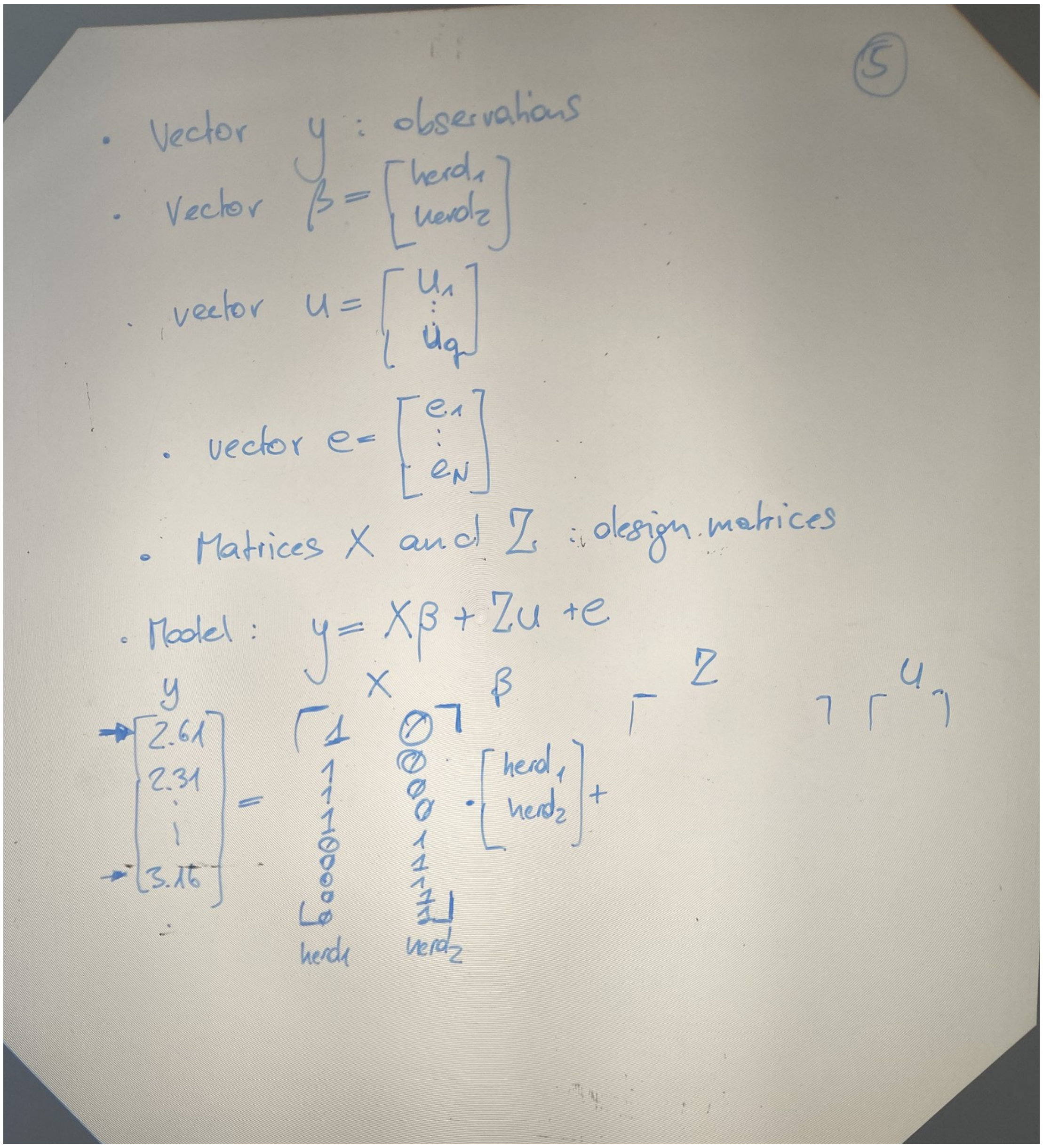
$$y_{ij} = \mu + \text{herd}_j + u_i + e_{ij}$$

- Dataset

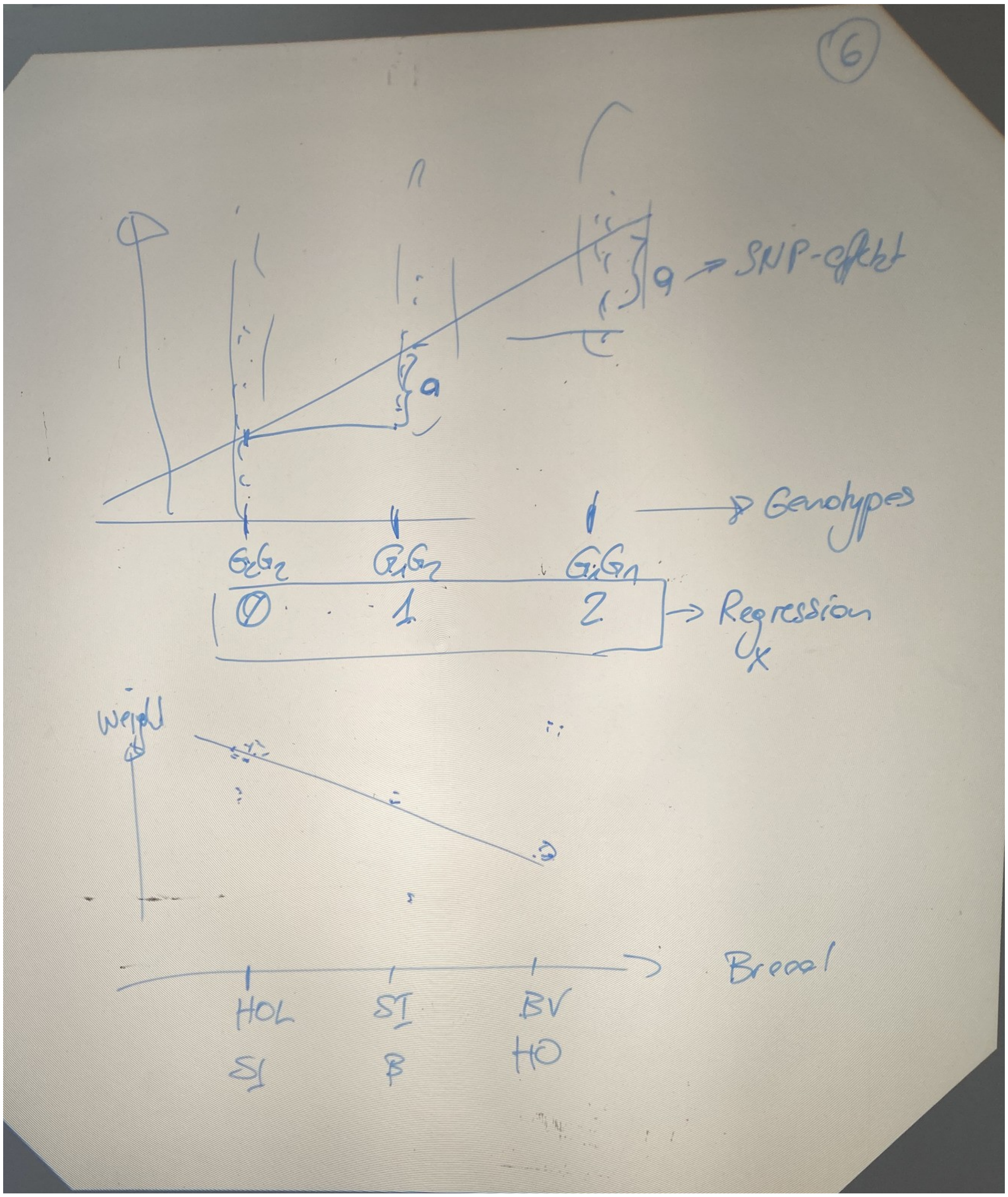
$$\begin{cases} y_{12,1} = \mu + \text{herd}_1 + u_{12} + e_{12,1} \\ y_{13,1} = \mu + \text{herd}_1 + u_{13} + e_{13,1} \\ \vdots \end{cases}$$

- Matrix-vector notation:
  - define vector $y = \begin{bmatrix} 2.61 \\ 2.31 \\ \vdots \\ 3.16 \end{bmatrix}$
  - define vector $\beta = \begin{bmatrix} \text{herd}_1 \\ \text{herd}_2 \end{bmatrix}$

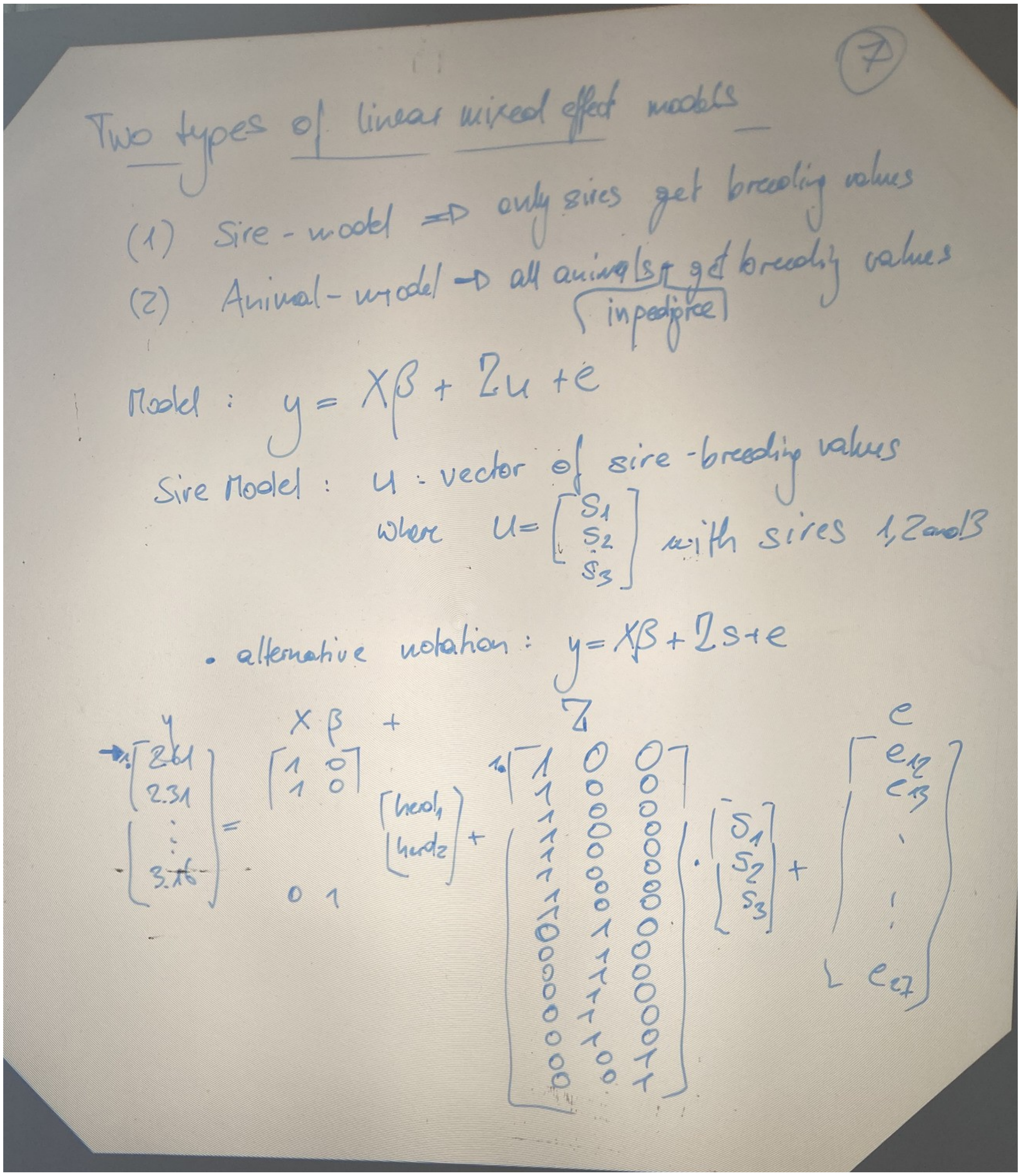
OHP Picture 5



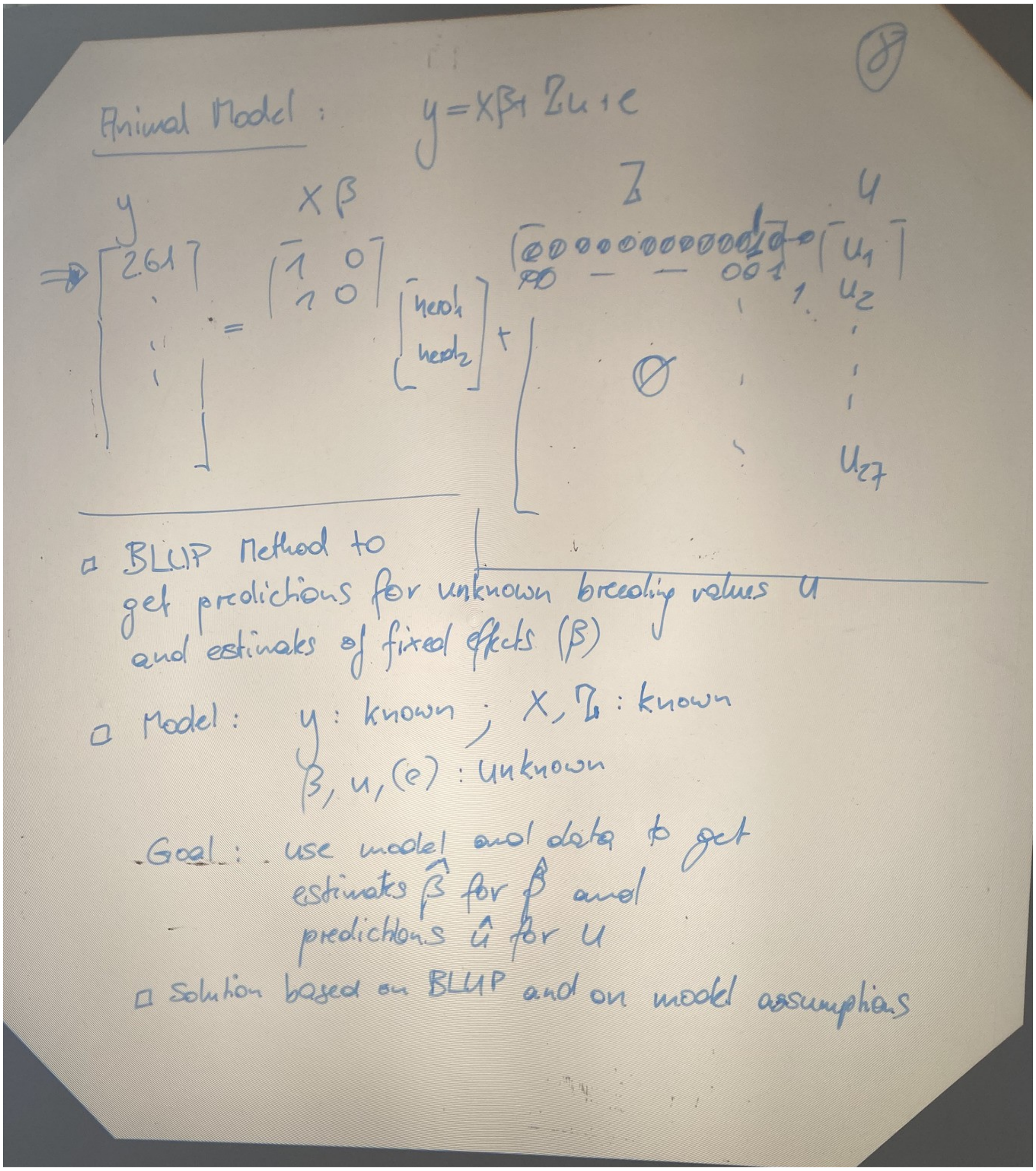
OHP Picture 6



OHP Picture 7



OHP Picture 8



OHP Picture 9

9

Model Assumptions: (must be specified for mixedmodel)

- Expected values
  - vector u of breeding values, defined as deviations $\Rightarrow E(u) = 0 \Leftrightarrow E\begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
  - vector e of residuals defined as deviations $\Rightarrow E(e) = 0$

$$\Rightarrow E(y) = E[X\beta + Zu + e]$$
$$= E(X\beta) + E(Zu) + \underbrace{E(e)}_{0}$$
$$= X\beta + Z \cdot \underbrace{E(u)}_{0}$$
$$= X\beta$$

- Abbreviation:

$$E\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} X\beta \\ 0 \\ 0 \end{bmatrix}$$

- Variance: $var(u) = G$; $var(e) = R$; $var(y) = V$
  where G and R are known variance-covariance matrices.
  $cov(u, e') = 0$

OHP Picture 10

(10)

- $var(u)$: variance-covariance matrix

$$G = var(u) = \begin{bmatrix} var(u_1) & cov(u_1,u_2) & cov(u_1,u_3) & \cdots \\ cov(u_2,u_1) & var(u_2) & cov(u_2,u_3) & \cdots \\ \vdots & & & \end{bmatrix}$$

where $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{27} \end{bmatrix}$

- $var(e) = R = \begin{bmatrix} var(e_1) & cov(e_1,e_2) & \cdots \\ cov(e_2,e_1) & var(e_2) & \cdots \end{bmatrix}$

$$var(e_1) = var(e_2) = \ldots = \sigma_e^2$$

$$cov(e_1,e_2) = cov(e_1,e_3) \ldots = \emptyset$$

$$\Rightarrow R = \begin{bmatrix} \sigma_e^2 & \emptyset & \emptyset & \cdots & \emptyset \\ \emptyset & \sigma_e^2 & & & \\ \vdots & & & & \end{bmatrix} = I \cdot \sigma_e^2$$

OHP Picture 11

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$$\mathrm{cov}(y,u) = \mathrm{cov}(X\beta + Zu + e, u^T)$$

$$= \underbrace{\mathrm{cov}(\underbrace{X\beta}_{fix}, u^T)}_{=\emptyset} + \mathrm{Cov}(Zu, u^T) + \underbrace{\mathrm{cov}(e, u^T)}_{=\emptyset}$$

$$=$$

$$= \mathrm{Cov}(Zu, u^T) = Z\underbrace{\mathrm{cov}(u, u^T)}_{var(u)} = ZG$$

$$\mathrm{cov}(y,e^T) = \mathrm{cov}(X\beta + Zu + e, e^T)$$

$$= \mathrm{cov}(X\beta, e^T) + \mathrm{cov}(Zu, e^T) + \mathrm{cov}(e, e^T)$$

$$= \emptyset + Z\underbrace{\mathrm{cov}(u, e^T)}_{=\emptyset} + \mathrm{var}(e)$$

$$= R$$

$$\mathrm{var}\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} V & ZG & R \\ GZ & G & \emptyset \\ R & \emptyset & R \end{bmatrix} ; \quad V = ZGZ^T + R$$

OHP Picture 12

12

- Using properties of BLUP:
  solutions for $\hat{u}$ and $\hat{\beta}$:

$$\hat{u} = G Z^T V^{-1} (y - X\hat{\beta})$$

$$\hat{\beta} = (X^T V^{-1} X)^{-} X^T V^{-1} y$$

- Practical Problem: Large data sets (10^7 Mio records)
  ($6 \cdot 10^7$ Mio)
  V^{-1}: cannot be computed

- Same solutions from system of equations:

$$\begin{bmatrix} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & Z^T R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{bmatrix}$$

- $R = I\sigma_e^2 \Rightarrow R^{-1} = I \cdot 1/\sigma_e^2$