OHP Picture 1

Recap. 2023-11-10

- Central topic:

  Use linear mixed effects (LME) to predict breeding values

- Why LME:

  Because breeding values of related individuals are correlated, for animals i and j sharing some common ancestor, we have

  $$cov(u_i, u_j) \neq \emptyset$$

- LME: In matrix-vector notation

$$\underbrace{y}_{\text{known vector of observations}} = X\underbrace{\beta}_{\text{unknown vector of fixed effects (herd, sex, breed, ...) or covariates such as breast circumference}} + Z\underbrace{u}_{\text{unknown vector of random breeding values}} + \underbrace{e}_{\text{unknown vector of random residuals}}$$

matrices: X, Z known design matrices

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□ Additional Specification / Assumptions for LME

- For each random effect, expected value and variance-covariance matrix must be specified

- Random effects are: u, e, y

$$\left.\begin{aligned} E(u) &= 0 \\ E(e) &= 0 \end{aligned}\right\} \text{ together with model } E(y) = E(X\beta + Zu + e) = X\beta$$

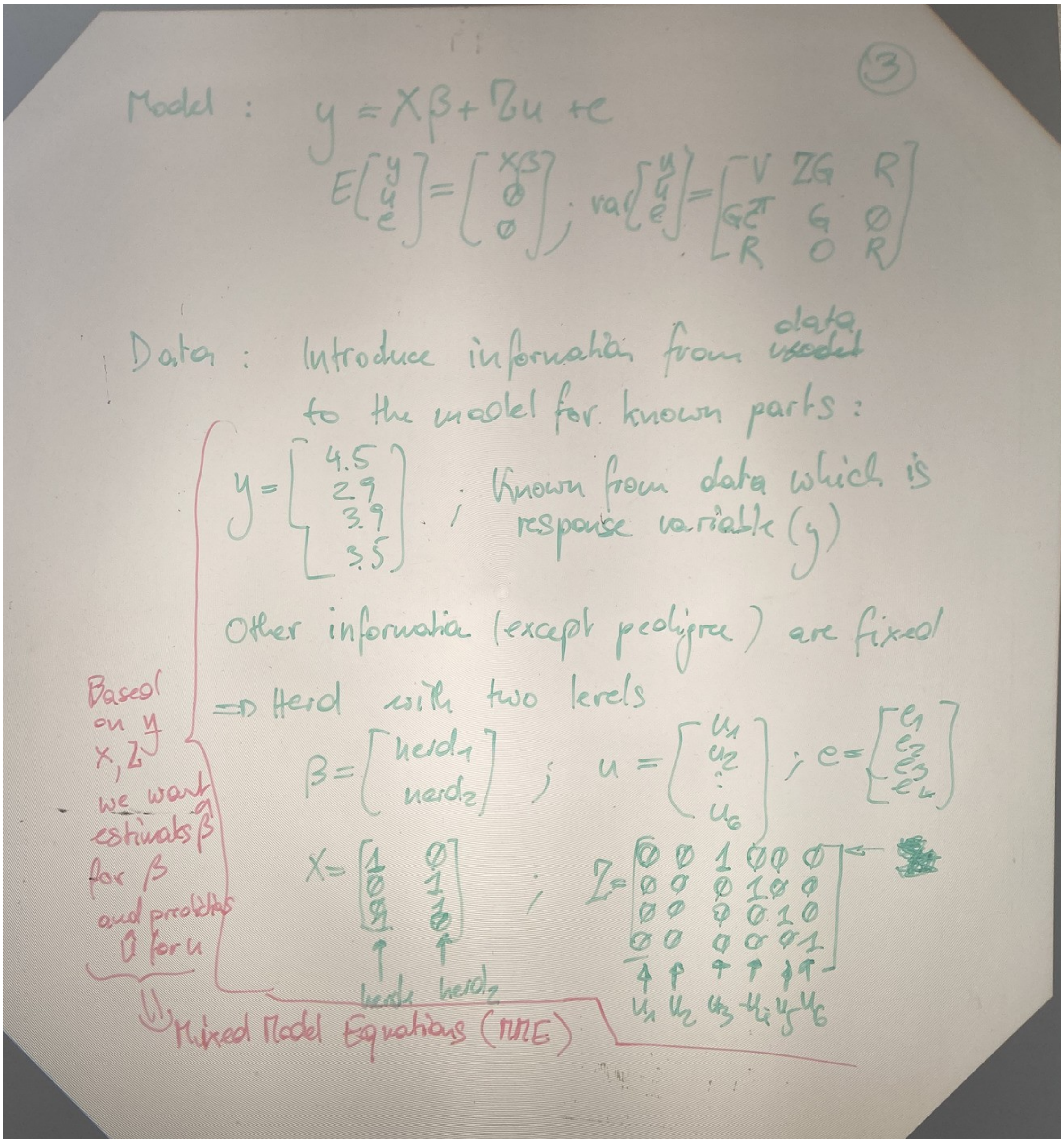
$$\left.\begin{aligned} var(u) &= G \\ var(e) &= R \end{aligned}\right\} var(y) = ZGZ^T + R = V$$

$$\left.\begin{aligned} cov(u, e^T) &= 0 \\ cov(y, u^T) &= ZG \\ cov(y, e) &= R \end{aligned}\right\} (*)$$

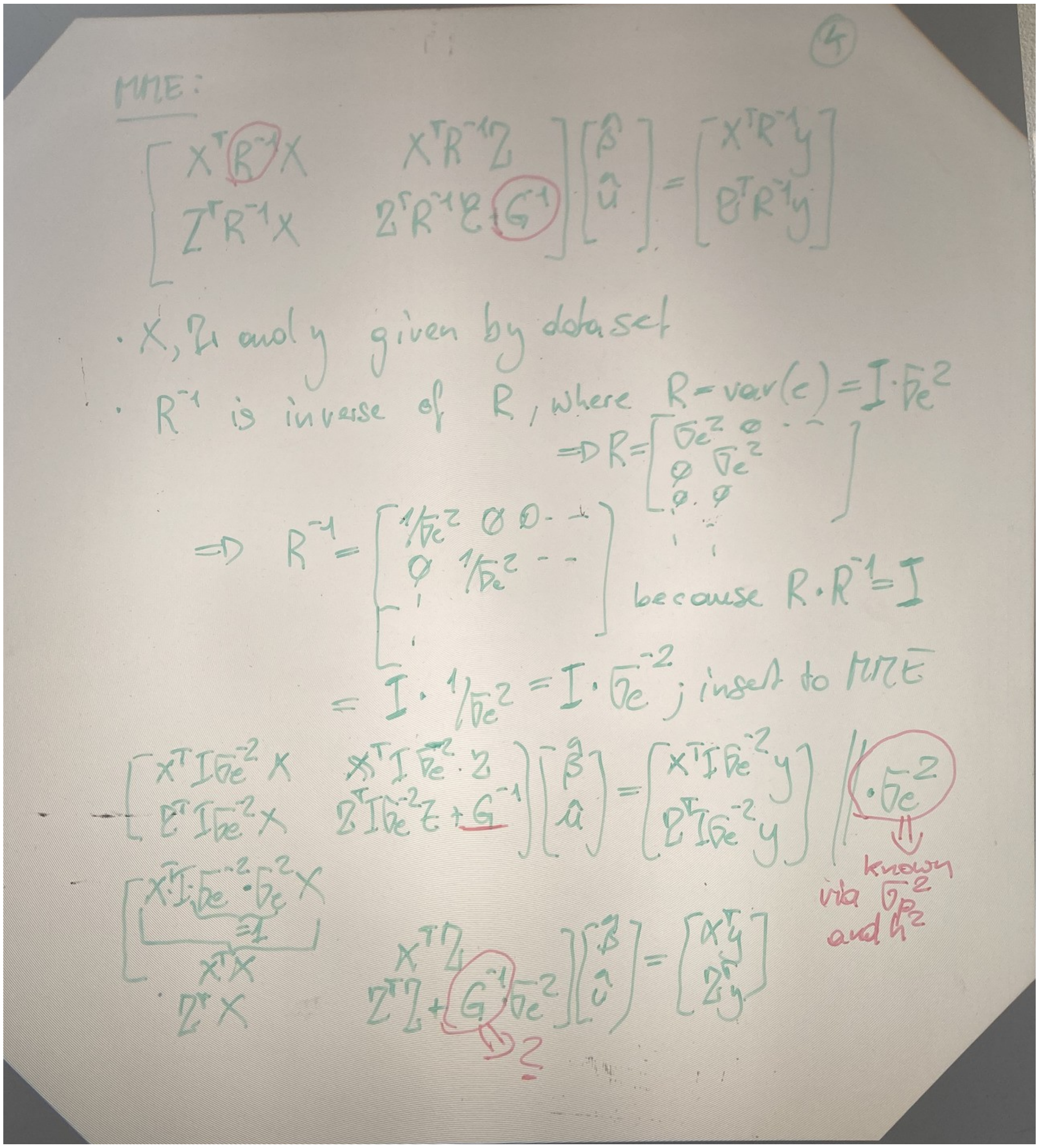
$$(*) \quad var\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} V & ZG & R \\ GZ^T & G & 0 \\ R & 0 & R \end{bmatrix}, \quad V = ZGZ^T + R$$

$$E\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} X\beta \\ 0 \\ 0 \end{bmatrix}$$

OHP Picture 3



OHP Picture 4



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MME:

$$\begin{bmatrix} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & Z^T R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{bmatrix}$$

- X, Z and y given by dataset
- R^{-1} is inverse of R, where $R = var(e) = I \cdot \sigma_e^2$

$$\Rightarrow R = \begin{bmatrix} \sigma_e^2 & 0 & \cdots \\ 0 & \sigma_e^2 & \\ 0 & 0 & \\ \vdots & \vdots & \end{bmatrix}$$

$$\Rightarrow R^{-1} = \begin{bmatrix} 1/\sigma_e^2 & 0 & 0 & \cdots \\ 0 & 1/\sigma_e^2 & \cdots & \\ \vdots & & & \\ \vdots & & & \end{bmatrix}$$ because $R \cdot R^{-1} = I$

$= I \cdot 1/\sigma_e^2 = I \cdot \sigma_e^{-2}$; insert to MME

$$\begin{bmatrix} X^T I \sigma_e^{-2} X & X^T I \sigma_e^{-2} \cdot Z \\ Z^T I \sigma_e^{-2} X & Z^T I \sigma_e^{-2} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T I \sigma_e^{-2} y \\ Z^T I \sigma_e^{-2} y \end{bmatrix} \Bigg|\Bigg| \cdot \sigma_e^2$$

$\Downarrow$ known via σ_P^2 and h^2

$$\underbrace{X^T I \cdot \sigma_e^{-2} \cdot \sigma_e^2}_{=I} X$$

$$\begin{bmatrix} X^T X & X^T Z \\ Z^T X & Z^T Z + G^{-1} \sigma_e^2 \end{bmatrix} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T y \\ Z^T y \end{bmatrix}$$

$G^{-1} \Rightarrow ?$

OHP Picture 5

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? What is G and G^{-1}

→ Specifikation of LME:

$var(u) = G$; example $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{bmatrix}$; if we use animal model

$$\Rightarrow var(u) = var\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{pmatrix} = \begin{bmatrix} var(u_1) & cov(u_1,u_2) & cov(u_1,u_6) \\ cov(u_2,u_1) & var(u_2) & \cdots \\ & & \ddots \end{bmatrix}$$

$$var(u_1) = \int (u_1 - E(u_1))^2 \cdot f(u_1)\, du_1$$

$$cov(u_1,u_2) = \iint (u_1 - E(u_1))(u_2 - E(u_2))\, f(u_1,u_2)\, du_1\, du_2$$

$$cov(u_1,u_1) = \iint (u_1 - E(u_1))(u_1 - E(u_1))\, f(u_1)\, du_1\, du_1$$

$$= \int (u_1 - E(u_1))^2 f(u_1)\, du_1 = var(u_1)$$

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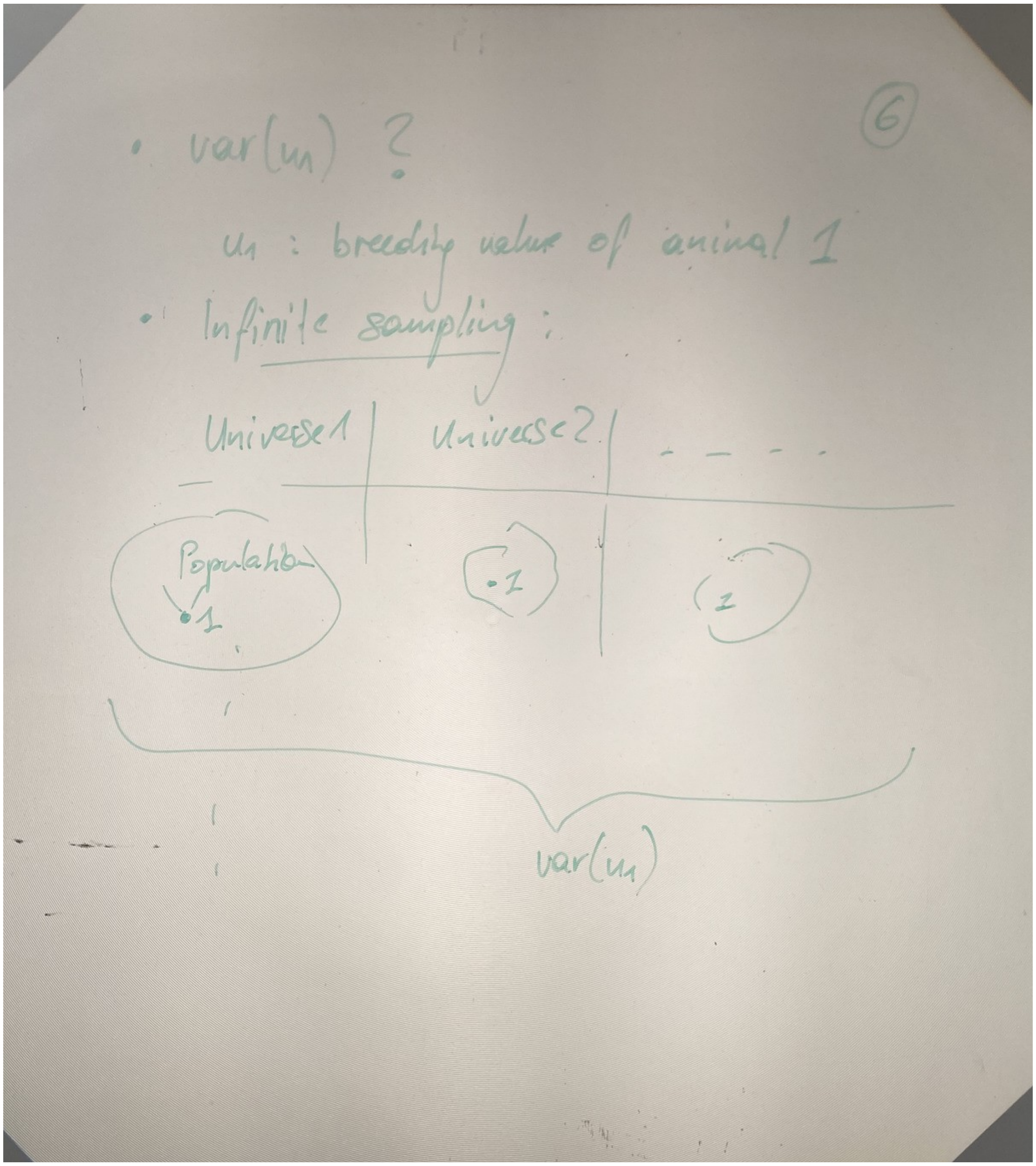
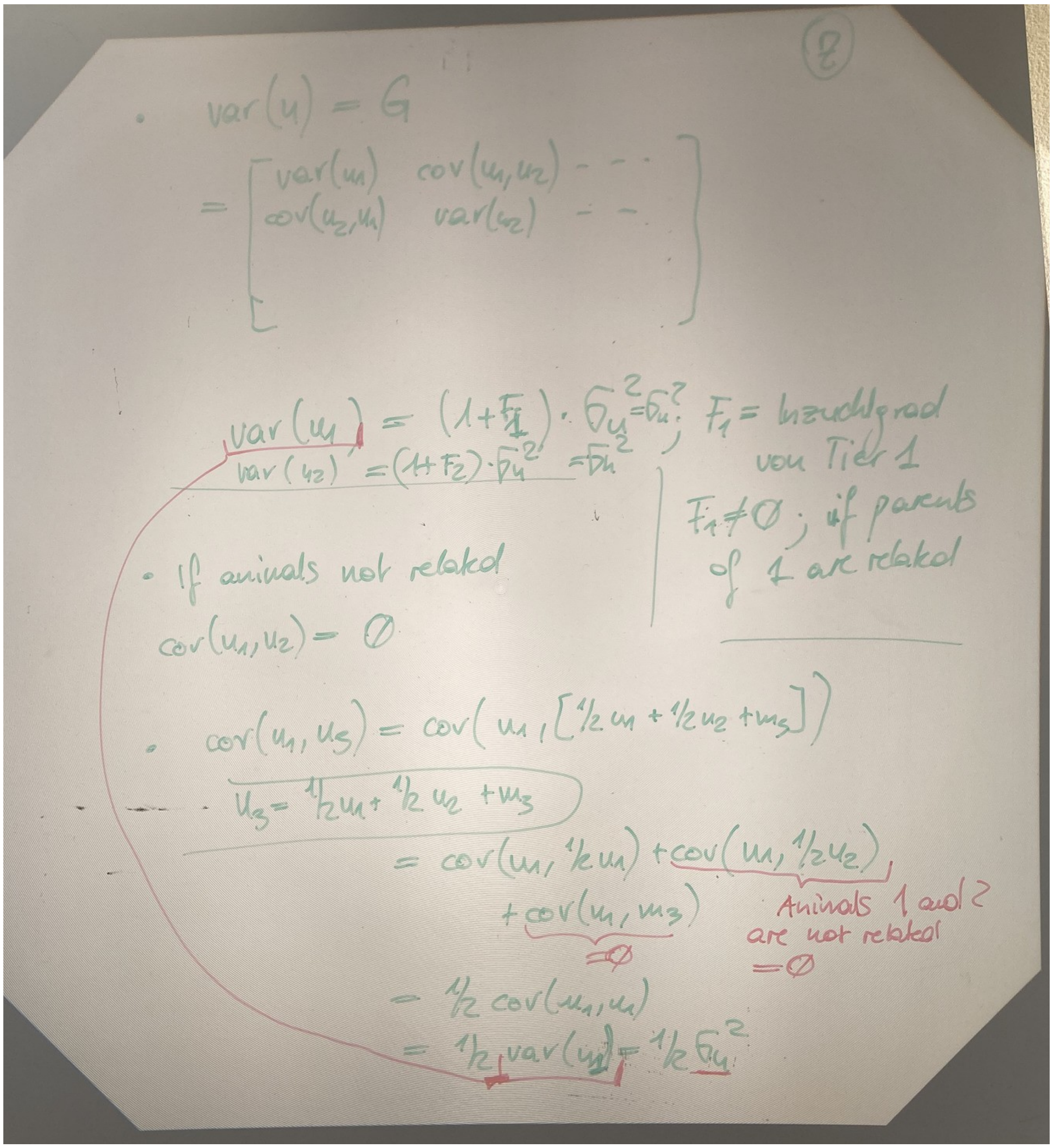


OHP Picture 7



(8)

- $\mathrm{var}(u) = G$

$$= \begin{bmatrix} \mathrm{var}(u_1) & \mathrm{cov}(u_1,u_2) & \cdots \\ \mathrm{cov}(u_2,u_1) & \mathrm{var}(u_2) & \cdots \\ \vdots & & \end{bmatrix}$$

$\mathrm{var}(u_1) = (1+F_1)\cdot\sigma_u^2 = \sigma_u^2$; F_1 = Inzuchtgrad von Tier 1

$\mathrm{var}(u_2) = (1+F_2)\cdot\sigma_u^2 = \sigma_u^2$

$F_1 \neq \emptyset$; if parents of 1 are related

- if animals not related

$\mathrm{cov}(u_1,u_2) = \emptyset$

- $\mathrm{cov}(u_1,u_3) = \mathrm{cov}(u_1, [\tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + m_3])$

$u_3 = \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + m_3$

$$= \mathrm{cov}(u_1, \tfrac{1}{2}u_1) + \underbrace{\mathrm{cov}(u_1, \tfrac{1}{2}u_2)}_{\text{Animals 1 and 2 are not related } = \emptyset} + \underbrace{\mathrm{cov}(u_1, m_3)}_{=\emptyset}$$

$$= \tfrac{1}{2}\,\mathrm{cov}(u_1,u_1)$$

$$= \tfrac{1}{2}\,\mathrm{var}(u_1) = \tfrac{1}{2}\sigma_u^2$$

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$$\operatorname{cov}(u_3, u_4) = \operatorname{cov}\left(\left[\tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + m_3\right], \left[\tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + m_4\right]\right)$$

$$\boxed{\begin{aligned} u_3 &= \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + m_3 \\ u_4 &= \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + m_4 \end{aligned}}$$

$$\begin{aligned}
&= \operatorname{cov}\left(\tfrac{1}{2}u_1, \tfrac{1}{2}u_1\right) + \overbrace{\operatorname{cov}\left(\tfrac{1}{2}u_1, \tfrac{1}{2}u_2\right)}^{=\emptyset} \\
&\quad + \overbrace{\operatorname{cov}\left(\tfrac{1}{2}u_1, m_4\right)}^{\emptyset} \\
&\quad + \overbrace{\operatorname{cov}\left(\tfrac{1}{2}u_2, \tfrac{1}{2}u_1\right)}^{=\emptyset} + \operatorname{cov}\left(\tfrac{1}{2}u_2, \tfrac{1}{2}u_2\right) \\
&\quad + \underbrace{\operatorname{cov}\left(\tfrac{1}{2}u_2, m_4\right)}_{=\emptyset} + \underbrace{\operatorname{cov}\left(m_3, \tfrac{1}{2}u_1\right)}_{0} \\
&\quad + \underbrace{\operatorname{cov}\left(m_3, \right)}_{0} \\
&\quad \underbrace{\operatorname{cov}\left(m_3, \right)}_{\emptyset} \\
&= \operatorname{cov}\left(\tfrac{1}{2}u_1, \tfrac{1}{2}u_1\right) + \operatorname{cov}\left(\tfrac{1}{2}u_2, \tfrac{1}{2}u_2\right) \\
&= \tfrac{1}{4}\operatorname{cov}(u_1, u_1) + \tfrac{1}{4}\operatorname{cov}(u_2, u_2) \\
&= \tfrac{1}{4}\operatorname{var}(u_1) + \tfrac{1}{4}\operatorname{var}(u_2) \\
&= \tfrac{1}{4}\sigma_u^2 + \tfrac{1}{4}\sigma_u^2 = \tfrac{1}{2}\sigma_u^2
\end{aligned}$$

OHP Picture 9

Summary: All elements of G depend on $\sigma_u^2 \Rightarrow G = \underbrace{A}_{\text{Numerator Relationship Matrix}} \cdot \sigma_u^2$

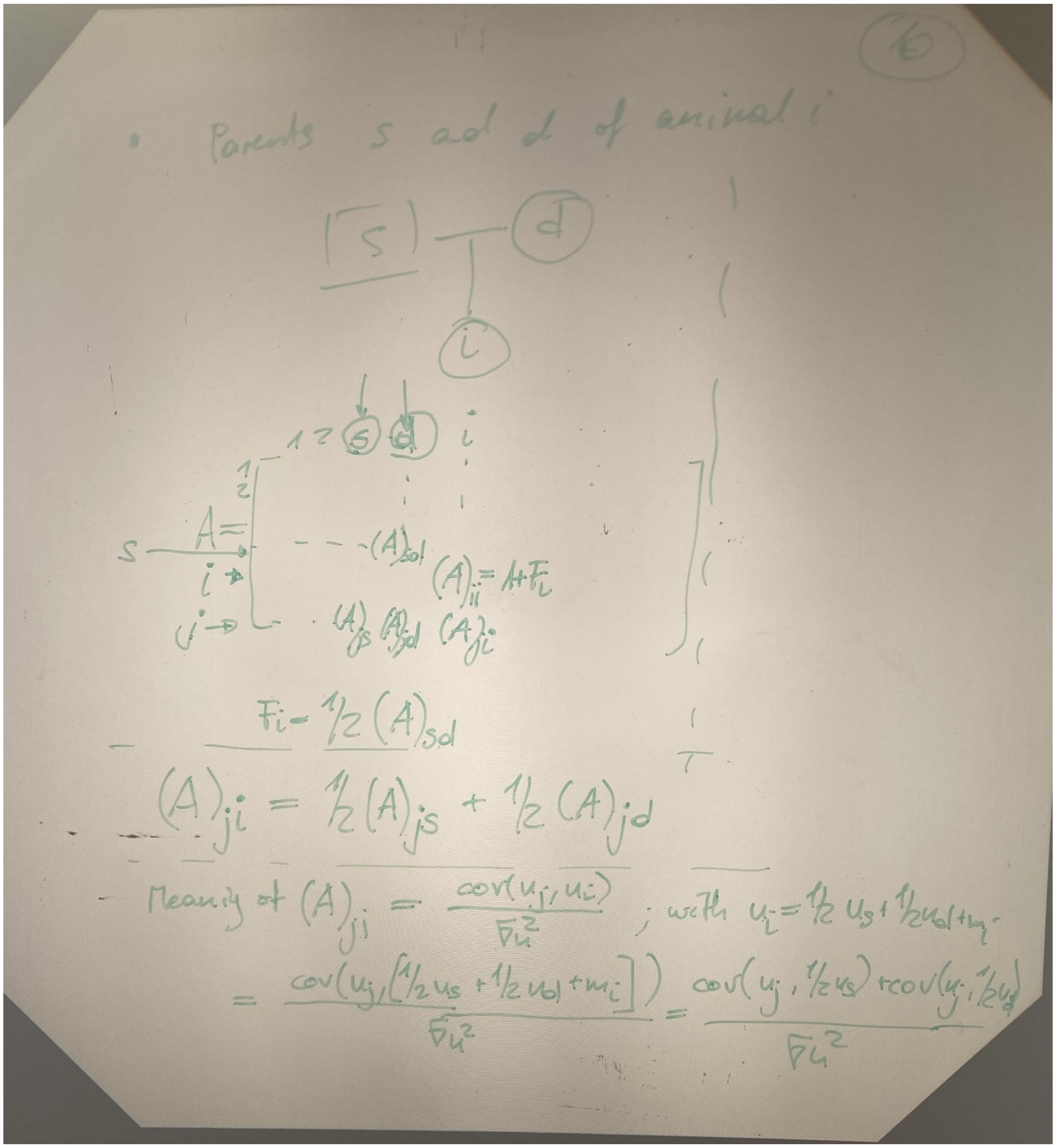
$\Rightarrow$ Definition of A:

- Diagonal elements: $(A)_{ii} = 1 + F_i$
- Off diagonal: $\underbrace{(A)_{ij}}_{\text{element in row } i \text{ and column } j} = \dfrac{cor(u_i, u_j)}{\sigma_u^2}$

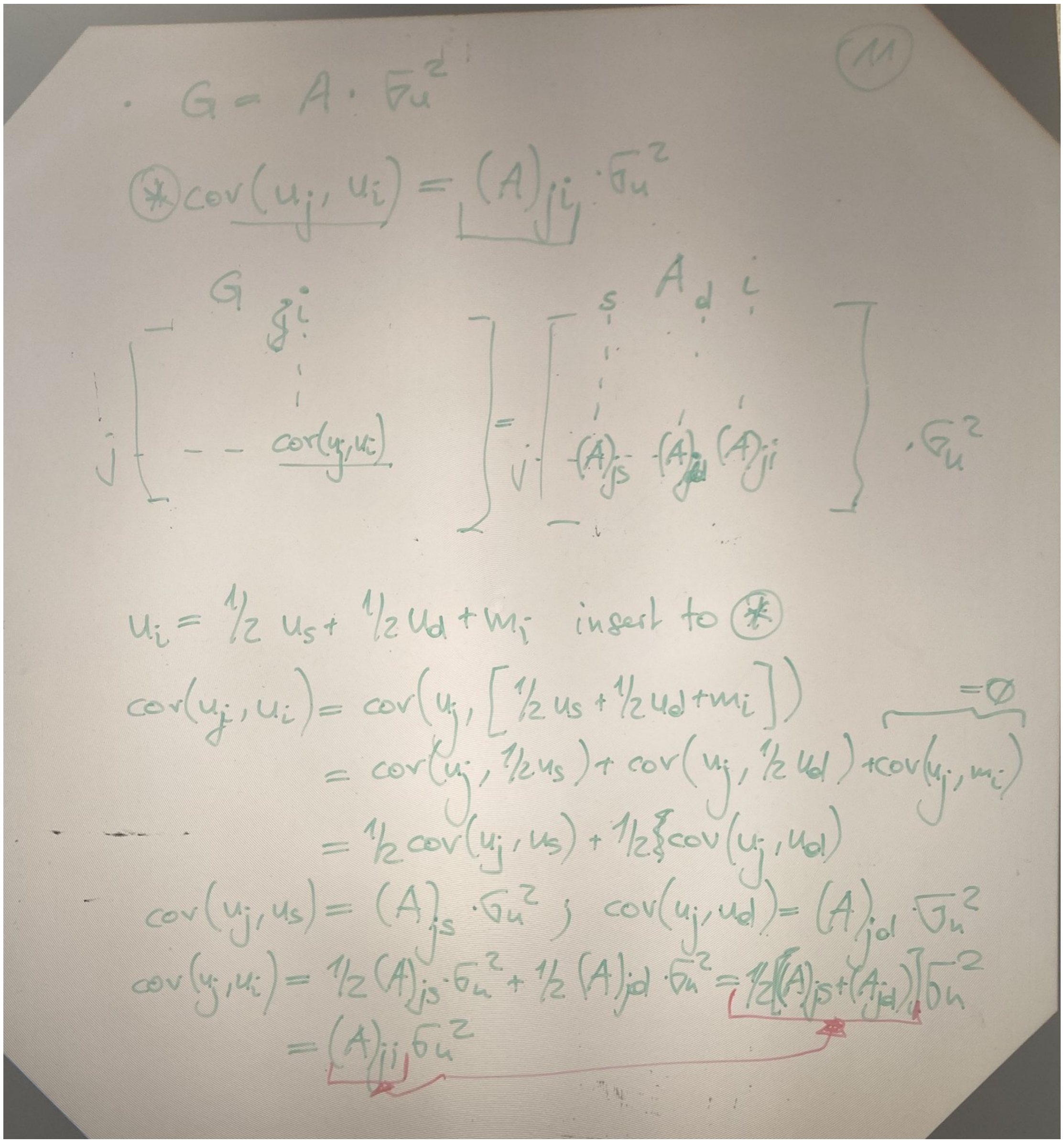
$$A = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & j \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & (A)_{ij} \end{bmatrix} \end{array}$$

$(A)_{ij}$ → relationship coefficient between animals i and j

OHP Picture 10



OHP Picture 11



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