

# OHP Picture 1

Recap: 2023-11-10 (6)

□ Central topic :  
Use linear mixed effects (LME) to predict breeding values

□ Why LME :  
Because breeding values of related individuals are correlated, for animals  $i$  and  $j$  sharing some common ancestor, we have  $\text{cov}(u_i, u_j) \neq 0$

□ LME : in matrix-vector notation

$$y = X\beta + Zu + e$$

$y$  → known vector of observations

$\beta$  → unknown vector of fixed effects (herd, sex, breed, ...) or covariates such as breast circumference

$u$  → unknown vector of random breeding values

$e$  → unknown vector of random residuals

matrices :  $X, Z$  known design matrices

OHP Picture 2

□ Additional Specification / Assumptions for LME ②

- For each random effect, expected value and variance-covariance matrix must be specified
- Random effects are:  $u, e, y$

$$\left. \begin{aligned} E(u) &= 0 \\ E(e) &= 0 \end{aligned} \right\} \text{together with model}$$

$$E(y) = E(X\beta + Zu + e) = X\beta$$

$$\left. \begin{aligned} \text{var}(u) &= G \\ \text{var}(e) &= R \end{aligned} \right\} \text{var}(y) = ZGZ^T + R = V$$

$$\left. \begin{aligned} \text{cov}(u, e^T) &= 0 \\ \text{cov}(y, u^T) &= ZG \\ \text{cov}(y, e) &= R \end{aligned} \right\} (*)$$


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(\*)  $\text{var} \begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} V & ZG & R \\ GZ^T & G & 0 \\ R & 0 & R \end{bmatrix}, V = ZGZ^T + R$

$$E \begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} X\beta \\ 0 \\ 0 \end{bmatrix}$$

OHP Picture 3

Model :  $y = X\beta + Zu + e$  ③

$$E \begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} X\beta \\ 0 \\ 0 \end{bmatrix}; \text{var} \begin{bmatrix} u \\ e \end{bmatrix} = \begin{bmatrix} -V & ZG & R \\ GZ^T & G & 0 \\ -R & 0 & R \end{bmatrix}$$

Data : Introduce information from <sup>data</sup> ~~model~~ to the model for known parts :

$$y = \begin{bmatrix} 4.5 \\ 2.9 \\ 3.9 \\ 3.5 \end{bmatrix}; \text{ known from data which is response variable } (y)$$

Other information (except pedigree) are fixed

⇒ herd with two levels

$$\beta = \begin{bmatrix} \text{herd}_1 \\ \text{herd}_2 \end{bmatrix}; u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{bmatrix}; e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; Z = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

↑            ↑

herd<sub>1</sub>    herd<sub>2</sub>

$u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6$

Based on  $y, X, Z$  we want estimates  $\beta$  for  $\beta$  and predict  $\hat{u}$  for  $u$

⇒ Mixed Model Equations (MME)

OHP Picture 4

④

MRE:

$$\begin{bmatrix} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & Z^T R^{-1} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{bmatrix}$$

- $X, Z$  only given by dataset
- $R^{-1}$  is inverse of  $R$ , where  $R = \text{var}(e) = I \cdot \sigma_e^2$   
 $\Rightarrow R = \begin{bmatrix} \sigma_e^2 & 0 & \dots \\ 0 & \sigma_e^2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$

$$\Rightarrow R^{-1} = \begin{bmatrix} 1/\sigma_e^2 & 0 & \dots \\ 0 & 1/\sigma_e^2 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \text{ because } R \cdot R^{-1} = I$$

$$= I \cdot 1/\sigma_e^2 = I \cdot \sigma_e^{-2}; \text{ insert to MRE}$$

$$\begin{bmatrix} X^T I \sigma_e^{-2} X & X^T I \sigma_e^{-2} Z \\ Z^T I \sigma_e^{-2} X & Z^T I \sigma_e^{-2} Z + G^{-1} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} X^T I \sigma_e^{-2} y \\ Z^T I \sigma_e^{-2} y \end{bmatrix} \quad \left| \begin{matrix} \cdot \sigma_e^{-2} \\ \downarrow \\ \text{known} \\ \text{via } \sigma_e^2 \\ \text{and } h^2 \end{matrix} \right.$$

$$\begin{bmatrix} X^T I \sigma_e^{-2} \cdot \sigma_e^2 X \\ X^T X \\ Z^T X \end{bmatrix} \begin{bmatrix} X^T Z \\ Z^T Z + G^{-1} \sigma_e^{-2} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} X^T y \\ Z^T y \end{bmatrix}$$

↓ ?

OHP Picture 5

2 What is  $G$  and  $G^{-1}$  ⑥

→ Specification of LME:

$\text{var}(u) = G$  ; example  $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{bmatrix}$ ; if

we use animal model

→  $\text{var}(u) = \text{var} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{pmatrix} = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \text{cov}(u_1, u_3) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$

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$\text{var}(u_1) = \int (u_1 - E(u_1))^2 \cdot f(u_1) \, du_1$

$\text{cov}(u_1, u_2) = \iint (u_1 - E(u_1))(u_2 - E(u_2)) f(u_1, u_2) \, du_1 \, du_2$

$\text{cov}(u_2, u_1) = \iint (u_1 - E(u_1))(u_2 - E(u_2)) f(u_1) \, du_1 \, du_2$   
 $= \int (u_1 - E(u_1))^2 f(u_1) \, du_1 = \text{var}(u_1)$

OHP Picture 6

•  $\text{var}(u_1)$  ? ⑥

$u_1$  : breeding value of animal 1

• Infinite sampling :

Universe 1	Universe 2	...
<p>Population • 1</p>	<p>• -1</p>	<p>• 2</p>

}  $\text{var}(u_1)$

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- $$\text{var}(u) = G$$

$$= \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$
- $$\frac{\text{var}(u_1)}{\text{var}(u_2)} = \frac{(1+F_1) \cdot \sigma_{u_1}^2}{(1+F_2) \cdot \sigma_{u_2}^2} = \frac{\sigma_{u_1}^2}{\sigma_{u_2}^2}$$

$F_1 = \text{Inzuchtgrad von Tier 1}$

$F_1 \neq 0$ ; if parents of 1 are related
- If animals not related

$$\text{cov}(u_1, u_2) = 0$$
- $$\text{cov}(u_1, u_3) = \text{cov}(u_1, [\frac{1}{2}u_1 + \frac{1}{2}u_2 + m_3])$$

$$u_3 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + m_3$$

$$= \text{cov}(u_1, \frac{1}{2}u_1) + \text{cov}(u_1, \frac{1}{2}u_2) + \text{cov}(u_1, m_3)$$

$\text{cov}(u_1, m_3) = 0$  Animals 1 and 2 are not related

$$= \frac{1}{2} \text{cov}(u_1, u_1)$$

$$= \frac{1}{2} \text{var}(u_1) = \frac{1}{2} \sigma_{u_1}^2$$

OHP Picture 8

⑧

$$\text{cov}(u_3, u_4) = \text{cov}\left[\left[\frac{1}{2}u_1 + \frac{1}{2}u_2 + u_3\right], \left[\frac{1}{2}u_1 + \frac{1}{2}u_2 + u_4\right]\right]$$

$$u_3 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + u_3$$

$$u_4 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + u_4$$

$$= \text{cov}\left(\frac{1}{2}u_1, \frac{1}{2}u_1\right) + \text{cov}\left(\frac{1}{2}u_1, \frac{1}{2}u_2\right) + \text{cov}\left(\frac{1}{2}u_1, u_4\right) + \text{cov}\left(\frac{1}{2}u_2, \frac{1}{2}u_1\right) + \text{cov}\left(\frac{1}{2}u_2, \frac{1}{2}u_2\right) + \text{cov}\left(\frac{1}{2}u_2, u_4\right) + \text{cov}\left(u_3, \frac{1}{2}u_1\right) + \text{cov}\left(u_3, \frac{1}{2}u_2\right) + \text{cov}\left(u_3, u_4\right)$$

$$= \text{cov}\left(\frac{1}{2}u_1, \frac{1}{2}u_1\right) + \text{cov}\left(\frac{1}{2}u_2, \frac{1}{2}u_2\right)$$

$$= \frac{1}{4} \text{cov}(u_1, u_1) + \frac{1}{4} \text{cov}(u_2, u_2)$$

$$= \frac{1}{4} \text{var}(u_1) + \frac{1}{4} \text{var}(u_2)$$

$$= \frac{1}{4} \sigma_u^2 + \frac{1}{4} \sigma_u^2 = \frac{1}{2} \sigma_u^2$$

OHP Picture 9

⑨

Summary: All elements of  $G$  depend on  $\bar{v}_u^2 \Rightarrow G = \underbrace{A}_{\substack{\text{Numerator Relationship} \\ \text{Matrix}}} \cdot \bar{v}_u^2$

$\Rightarrow$  Definition of  $A$ :

- Diagonal elements:  $(A)_{ii} = 1 + F_i$
- Off diagonal:  $(A)_{ij} = \frac{\text{cov}(u_i, u_j)}{\bar{v}_u^2}$   
 element in row  $i$  and column  $j$

$A = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \dots & & \\ & & & & j \\ & & & & & \vdots \\ & & & & & & i \\ & & & & & & & \vdots \\ & & & & & & & & \vdots \end{bmatrix}$   
 $\rightarrow$  relationship coefficient between animals  $i$  and  $j$

OHP Picture 10

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• Parents s and d of animal i

$$F_i = \frac{1}{2}(A)_{sd}$$

$$(A)_{ji} = \frac{1}{2}(A)_{js} + \frac{1}{2}(A)_{jd}$$

Mean of  $(A)_{ji} = \frac{\text{cov}(u_j, u_i)}{F_i^2}$ ; with  $u_i = \frac{1}{2}u_s + \frac{1}{2}u_d + m_i$

$$= \frac{\text{cov}(u_j, [\frac{1}{2}u_s + \frac{1}{2}u_d + m_i])}{F_i^2} = \frac{\text{cov}(u_j, \frac{1}{2}u_s) + \text{cov}(u_j, \frac{1}{2}u_d)}{F_i^2}$$

OHP Picture 11

(11)

$G = A \cdot \sigma_u^2$

$\textcircled{*} \text{cov}(u_j, u_i) = (A)_{ji} \cdot \sigma_u^2$

$$G \begin{bmatrix} g_{11} \\ \vdots \\ g_{jj} \\ \vdots \\ \text{---} \frac{\text{cov}(u_j, u_i)}{\text{---}} \end{bmatrix} = \begin{bmatrix} s & & & & A_{d1} \\ & \ddots & & & \vdots \\ & & & & (A)_{js} \\ & & & & (A)_{jd} \\ & & & & (A)_{ji} \end{bmatrix} \cdot \sigma_u^2$$

$u_i = \frac{1}{2} u_s + \frac{1}{2} u_d + m_i$  insert to  $\textcircled{*}$

$$\begin{aligned} \text{cov}(u_j, u_i) &= \text{cov}(u_j, [\frac{1}{2} u_s + \frac{1}{2} u_d + m_i]) \\ &= \text{cov}(u_j, \frac{1}{2} u_s) + \text{cov}(u_j, \frac{1}{2} u_d) + \underbrace{\text{cov}(u_j, m_i)}_{=0} \\ &= \frac{1}{2} \text{cov}(u_j, u_s) + \frac{1}{2} \text{cov}(u_j, u_d) \end{aligned}$$

$\text{cov}(u_j, u_s) = (A)_{js} \cdot \sigma_u^2$  ;  $\text{cov}(u_j, u_d) = (A)_{jd} \cdot \sigma_u^2$

$$\begin{aligned} \text{cov}(u_j, u_i) &= \frac{1}{2} (A)_{js} \cdot \sigma_u^2 + \frac{1}{2} (A)_{jd} \cdot \sigma_u^2 = \frac{1}{2} [(A)_{js} + (A)_{jd}] \sigma_u^2 \\ &= (A)_{ji} \sigma_u^2 \end{aligned}$$

OHP Picture 12

12

3	①	2	-3	12
4	1	NA	4	1 NA
5	4	3	5	4 3
6	5	2	6	5 2

→

→ 1.	NA NA
2	NA NA

$$A = \begin{matrix} & \begin{matrix} NA & NA & NA & NA & 2 & 1 & NA & 4 & 3 & 5 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{1/2} & \boxed{1/2} & \boxed{1/2} & \boxed{1/4} \\ \boxed{0} & \boxed{1} & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{matrix}$$

$(A)_{11} = 1 + F_1 = 1 + 0 = 1$

$(A)_{12} = \frac{1}{2}(A)_{1NA} + \frac{1}{2}(A)_{1NA} = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$

$(A)_{13} = \frac{1}{2}(A)_{11} + \frac{1}{2}(A)_{12} = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$

$(A)_{14} = \frac{1}{2}(A)_{11} + 0 = \frac{1}{2}$

$(A)_{15} = \frac{1}{2}(A)_{14} + \frac{1}{2}(A)_{13} = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

$(A)_{16} = \frac{1}{2}(A)_{15} + \frac{1}{2}(A)_{12} = \frac{1}{2} \cdot \frac{1}{2} + 0 = \frac{1}{4}$