OHP Picture 1

1

Recap 2023-12-01

- Prediction of breeding values using BLUP-animal model
- BLUP animal model: Linear mixed effects model (LME)

$$y = X\beta + Zu + e$$

; with β: fixed effects

u: breeding values

y: response variable or observations

e: residuals

- BLUE-estimates $\hat{\beta}$ for fixed effects β
  BLUP-predictions $\hat{u}$ for breeding values u
- Mixed model equations:

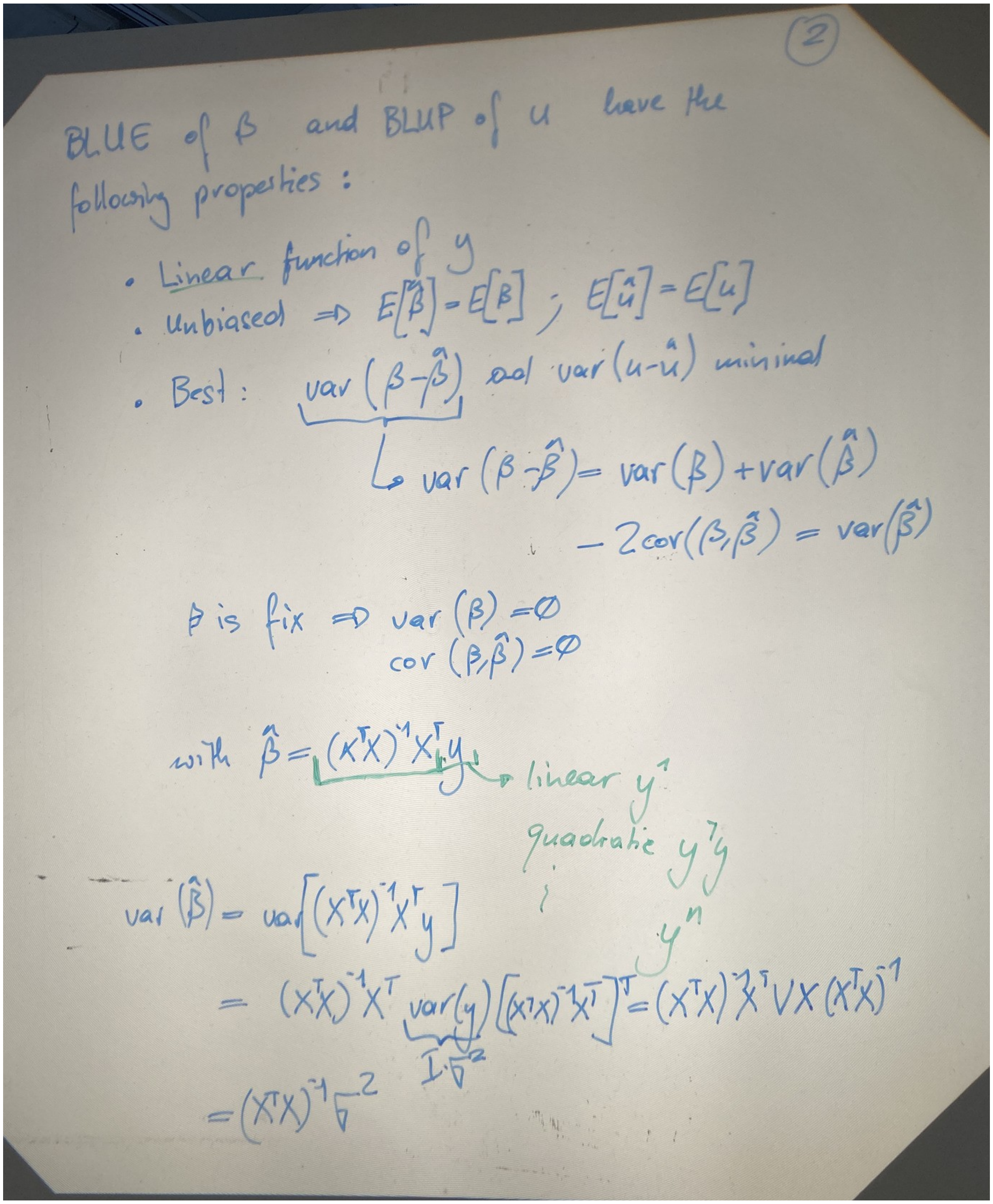
$$\underbrace{\begin{bmatrix} X^TX & X^TZ \\ Z^TX & Z^TZ + \lambda A^{-1} \end{bmatrix}}_{M} \cdot \underbrace{\begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix}}_{\hat{s}} = \underbrace{\begin{bmatrix} X^Ty \\ Z^Ty \end{bmatrix}}_{r}$$

; $\lambda = \frac{\sigma_e^2}{\sigma_u^2}$

A^{-1}: inverse numerator relationship matrix

$$\hat{s} = \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = M^{-1} \cdot r \Rightarrow \text{solve}(M, r) \text{ in R}$$

OHP Picture 2



2

BLUE of β and BLUP of u have the following properties:

- Linear function of y
- Unbiased $\Rightarrow E[\hat{\beta}] = E[\beta]$; $E[\hat{u}] = E[u]$
- Best: $\underbrace{var(\beta - \hat{\beta})}$ and $var(u - \hat{u})$ minimal

$$\hookrightarrow var(\beta - \hat{\beta}) = var(\beta) + var(\hat{\beta}) - 2cor((\beta, \hat{\beta}) = var(\hat{\beta})$$

β is fix $\Rightarrow var(\beta) = \emptyset$

$cor(\beta, \hat{\beta}) = \emptyset$

with $\hat{\beta} = \underbrace{(X^TX)^{-1}X^T y}$ $\rightarrow$ linear y^1

quadratic $y^T y$

y^n

$$var(\hat{\beta}) = var[(X^TX)^{-1}X^T y]$$

$$= (X^TX)^{-1}X^T \underbrace{var(y)}_{I \cdot \sigma^2} [(X^TX)^{-1}X^T]^T = (X^TX)^{-1}X^T V X (X^TX)^{-1}$$

$$= (X^TX)^{-1}\sigma^2$$

OHP Picture 3

3

- BLUP of u

  $var(u - \hat{u})$ is called Prediction Error Variance (PEV)

- u is random $\Rightarrow$ $var(u) = G$

- Compute $var(u - \hat{u}) = var(u) + var(\hat{u}) - 2Cor(u, \hat{u})$

- Based on properties of BLUP

  $var(\hat{u}) = cov(u, \hat{u})$ insert in PEV

  where $\hat{u} = G \cdot Z \cdot V^{-1}(y - X\beta)$

$\Rightarrow$ $PEV = var(u - \hat{u}) = var(u) + var(\hat{u}) - 2cov(u, \hat{u})$

$= var(u) - var(\hat{u})$

- PEV is related to reliability (B% − Bestimmtheitsmass)

  $B = r^2_{u,\hat{u}}$ with $B\% = 100 \cdot B$

OHP Picture 4

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- $PEV(\hat{u}) = var(u) - var(\hat{u}) = C^{22}$
- C^{22} is part of inverse of coefficient Matrix M of mixed model equations
- $$\underbrace{\begin{bmatrix} X^TR^{-1}X & X^TR^{-1}Z \\ Z^TR^{-1}X & Z^TR^{-1}Z + G^{-1} \end{bmatrix}}_{M} \begin{bmatrix} \hat{\beta} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^TR^{-1}y \\ Z^TR^{-1}y \end{bmatrix}$$

$$\hat{s} = r$$

$$M^{-1} = \begin{bmatrix} X^TR^{-1}X & X^TR^{-1}Z \\ Z^TR^{-1}X & \underbrace{Z^TR^{-1}Z + G}_{\substack{\text{Anzahl col} \,\hat{=} \\ \text{Number of animals in pedigree}}} \end{bmatrix}^{-1} = \begin{bmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{bmatrix}$$

C^{22} is symmetric with dimensions $q \times q$ with q being the number of animals in Pedigree.

OHP Picture 5

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□ Reliability of predicted breeding value $\hat{u}_i$ for animal i:

$$B_i = r^2_{u_i,\hat{u}_i} = \frac{cov(u_i,\hat{u}_i)^2}{var(u_i)\, var(\hat{u}_i)} = \frac{var(\hat{u}_i)^2}{var(u_i)\, var(\hat{u}_i)}$$

using BLUP property : $var(\hat{u}_i) = cov(u_i, \hat{u}_i)$

$$B_i = \frac{var(\hat{u}_i)}{var(u_i)} \Rightarrow var(\hat{u}_i) = B_i \cdot var(u_i)$$

$$PEV(u_i) = var(u_i) - var(\hat{u}_i)$$

$$\underbrace{= var(u_i) - B_i\, var(u_i)}_{\text{solve for } B_i} = (1 - B_i)\, var(u_i)$$

$$B_i\, var(u_i) = var(u_i) - PEV(u_i)$$

$$B_i = \frac{var(u_i) - PEV(u_i)}{var(u_i)} = 1 - \frac{PEV(u_i)}{var(u_i)}$$

$$= 1 - \frac{(G^{22})_{ii}}{var(u_i)}$$

OHP Picture 6

(6)

□ MME

$$\begin{bmatrix} X^TX & X^TZ \\ Z^TX & Z^TZ+\lambda A^{-1}\end{bmatrix}\begin{bmatrix}\hat{\beta}\\ \hat{u}\end{bmatrix} = \begin{bmatrix} X^Ty \\ Z^Ty\end{bmatrix}$$

$$\lambda = \sigma_e^2 / \sigma_u^2$$

$*\sigma_e^{-2}$ mat_xtx

$$\square \quad \begin{bmatrix} X^TR^{-1}X & X^TR^{-1}Z \\ Z^TR^{-1}X & \underbrace{Z^TR^{-1}Z+G^{-1}}_{C^{22}}\end{bmatrix}$$

$$R^{-1} = I\cdot\sigma_e^{-2}$$

$$G^{-1} = A^{-1}\cdot\sigma_u^{-2}$$

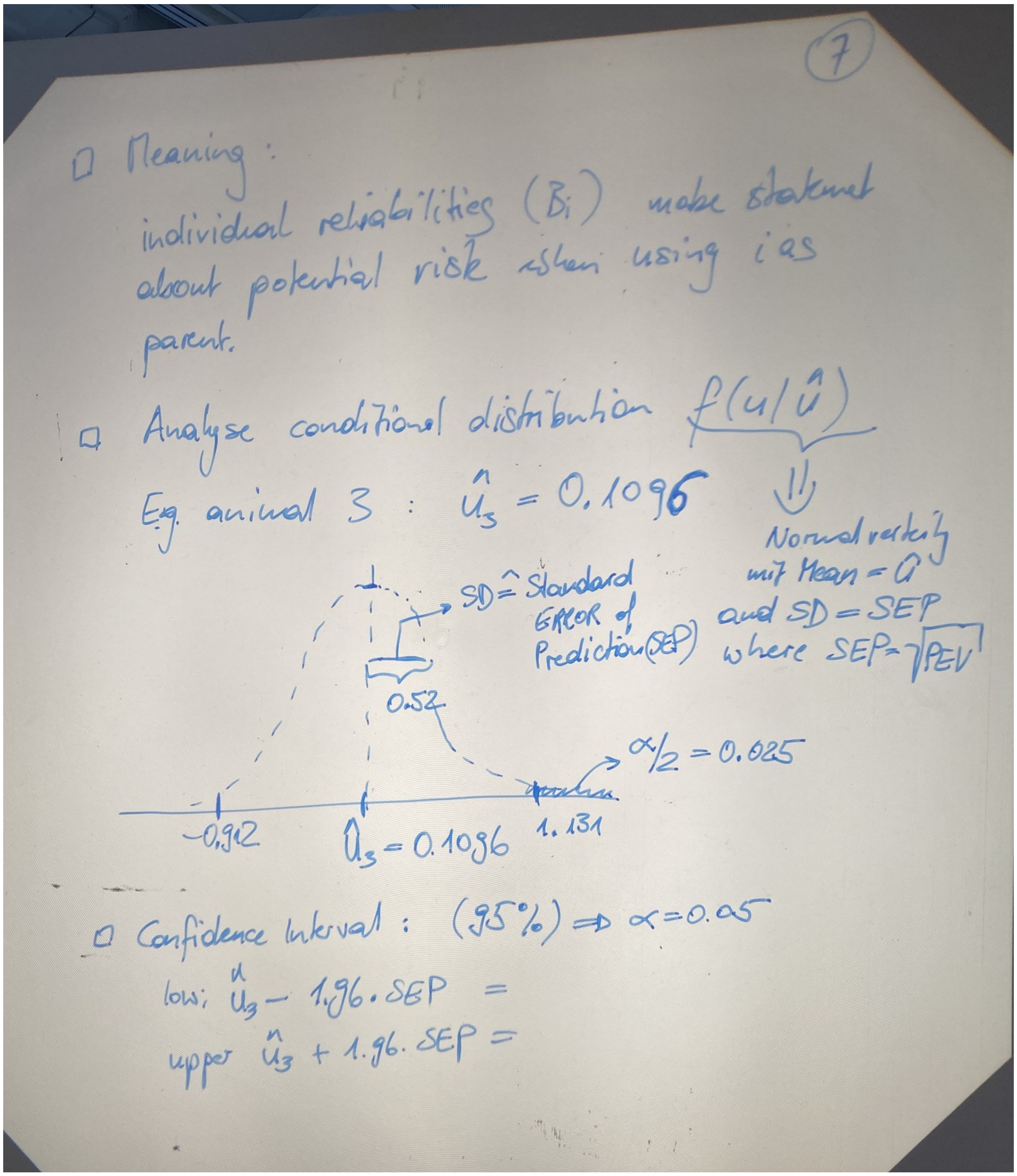
$$\lambda = \sigma_e^2/\sigma_u^2 = 1 \quad ; \quad \sigma_p^2 = 0.65 = \sigma_e^2 + \sigma_u^2$$

$$\sigma_u^2 = \sigma_p^2/2 =$$

$$h^2 = \frac{\sigma_u^2}{\sigma_p^2} = \frac{\sigma_u^2}{\sigma_u^2+\sigma_e^2} = 0.5$$

$$B_i = 1 - \frac{(C^{22})_{ii}}{var(u_i)}$$

OHP Picture 7



OHP Picture 8

(8)

- Selection Response (R)

$$R = i \cdot r_{u,\hat{u}} \cdot \sigma_u$$

- For a breeding program:
  - R per generation interval $= R/L$
  - 4-path model

$$\begin{tikzcd}
\text{Sire} \arrow[d, "1"] \arrow[dr, "2"] & \text{Dam} \arrow[dl, "3"'] \arrow[d, "4"] \\
\text{Sire} & \text{Dam}
\end{tikzcd}$$

$$R/L = \frac{R_1 + R_2 + R_3 + R_4}{L_1 + L_2 + L_3 + L_4}$$

→ generation interval on each path

$$R_1 = i_1 \, r_{u,\hat{u}} \cdot \sigma_u$$

→ accuracy for all male selection candidates not individual accuracies.

$R_2 =$

$R_3 =$

$R_4 =$

OHP Picture 9