# Additional Aspects of BLUP

Peter von Rohr

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# Aspects

- Accurracy
  - Results from MME are estimates of fixed effects and predictions of breeding values
  - Need statement about quality of estimates and predictions
- Confidence Intervals
- Decomposition of Predicted Breeding values

# Accurracy - Fixed Effects

- One property of BLUP was that variance of prediction error is minimal
- How can we measure the variance of the prediction error
- Fixed effects

$$var(\beta - \hat{\beta}) = var(\hat{\beta})$$

- Reminder:

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

with $X$ having full column rank

# Accurracy - Random effects

$$var(u - \hat{u}) = var(u) - 2 * cov(u, \hat{u}) + var(\hat{u}) = var(u) - var(\hat{u}) = PEV(\hat{u})$$

because with BLUP: $cov(u, \hat{u}) = var(\hat{u})$

# PEV

- PEV depends on inverse of coefficient matrix of MME

$$\left[\begin{array}{lr} X^TR^{-1}X & X^TR^{-1}Z \\ Z^TR^{-1}X & Z^TR^{-1}Z + G^{-1} \end{array}\right]^{-1} = \left[\begin{array}{cc} C^{11} & C^{12} \\ C^{21} & C^{22} \end{array}\right]$$

- For predicted breeding values $\hat{u}$

$$PEV(\hat{u}) = var(u) - var(\hat{u}) = C^{22}$$

# Single Animal $i$

$$PEV(\hat{u}_i) = (C)^{22}_{ii}$$

where $(C)^{22}_{ii}$ is the $i$-th diagonal of $C^{22}$

- Accuracy measured by correlation

$$r_{u_i,\hat{u}_i} = \frac{cov(u_i, \hat{u}_i)}{\sqrt{var(u_i) * var(\hat{u}_i)}} = \sqrt{\frac{var(\hat{u}_i)}{var(u_i)}}$$

- Combining

$$PEV(\hat{u}_i) = (C)^{22}_{ii} = var(u_i) - var(\hat{u}_i) = var(u_i) - r^2_{u_i,\hat{u}_i} var(u_i)$$

# Reliability $B_i$

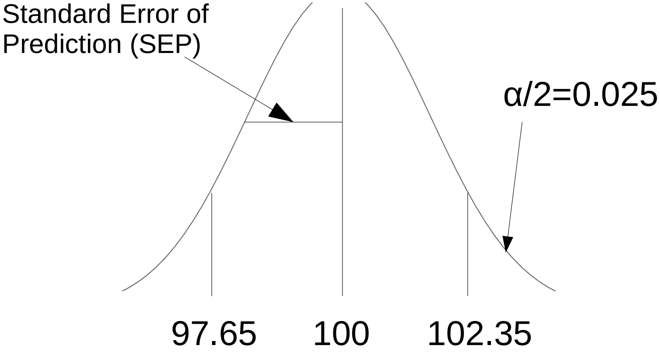
$$B_i = r^2_{u_i,\hat{u}_i} = \frac{var(u_i) - (C)^{22}_{ii}}{var(u_i)} = 1 - \frac{PEV(\hat{u}_i)}{var(u_i)} = 1 - \frac{(C)^{22}_{ii}}{var(u_i)}$$

- $B_i$ is large for small $PEV(\hat{u}_i)$
- In the limit $B_i \rightarrow 1$ for $PEV(\hat{u}_i) \rightarrow 0$
- For $PEV(\hat{u}_i) \rightarrow 0$ we must have $var(\hat{u}_i) \rightarrow var(u_i)$
- Therefore, the closer $var(\hat{u}_i)$ is to $var(u_i)$, the more accurate the predicted breeding value

# Confidence Intervals of $\hat{u}_i$

- Predicted breeding value ($\hat{u}_i$) is a function of the data ($y$)
- Hence $\hat{u}_i$ is a random variable with a distribution

# Distribution




$$SEP(\hat{u}_i) = \sqrt{PEV(\hat{u}_i)} = \sqrt{(1 - r^2_{u_i,\hat{u}_i}) * var(u_i)}$$

# Widths Of Confidence Intervals

Table 1: Widths of Confidence Intervals for Given Accuracies

| Accurracy | Interval Width |
|---:|---:|
| 0.40 | 36.44 |
| 0.50 | 33.26 |
| 0.60 | 29.75 |
| 0.70 | 25.76 |
| 0.80 | 21.04 |
| 0.90 | 14.88 |
| 0.95 | 10.52 |
| 0.99 | 4.70 |

with $\hat{u}_i = 100$, $var(u_i) = 144$ and $\alpha = 0.05$

# Selection Response

Correlation $r_{u_i,\hat{u}_i}$ for a single animal $i$

- across conceptual repeated sampling
- change of a predicted breeding value for animal $i$ with increasing information
- related to standard error of prediction (SEP) $\rightarrow$ measure of risk of using $i$ as parent

Accuracies also important for selection response

- correlation between true and predicted breeding values in selection candidates
- characteristic of population not of single animal
- estimation with cross-validation

$\rightarrow$ use correlation between predicted breeding values with whole and partial data

# Decomposition of Predicted Breeding Value

- Write MME as

$$M \cdot s = r$$

with

$$s = \left[ \begin{array}{c} \hat{\beta} \\ \hat{u} \end{array} \right]$$

- $\hat{\beta}$ has length $p$
- $\hat{u}$ has length $q$

# Simplified Model

$$y_i = \mu + u_i + e_i$$

where

| | |
|---|---|
| $y_i$ | Observation for animal $i$ |
| $u_i$ | breeding value of animal $i$ with a variance of $(1 + F_i)\sigma_u^2$ |
| $e_i$ | random residual effect with variance $\sigma_e^2$ |
| $\mu$ | single fixed effect |

# Data

- all animals have an observation
- animal $i$ has
  - parents $s$ and $d$
  - $n$ progeny $k_j$ (with $j = 1, \ldots, n$)
  - $n$ mates $l_j$ (with $j = 1, \ldots, n$).
- progeny $k_j$ has parents $i$ and $l_j$.

# Example

| Animal | Sire | Dam | WWG |
|---|---|---|---|
| 1 | NA | NA | 4.5 |
| 2 | NA | NA | 2.9 |
| 3 | NA | NA | 3.9 |
| 4 | 1 | 2 | 3.5 |
| 5 | 4 | 3 | 5.0 |

Variance components $\sigma_e^2 = 40$ and $\sigma_u^2 = 20$.

# Model Components

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; Z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 5 \end{bmatrix}, \; X^T Z = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$Z^T Z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Right-hand Side

$$X^T y = \left[ \sum_{j=1}^{n} y_i \right] = 19.8$$

$$Z^T y = \begin{bmatrix} y1 \\ y2 \\ y3 \\ y4 \\ y5 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 2.9 \\ 3.9 \\ 3.5 \\ 5 \end{bmatrix}$$

# $A^{-1}$

$$A^{-1} = \begin{bmatrix} 1.5 & 0.5 & 0 & -1 & 0 \\ 0.5 & 1.5 & 0 & -1 & 0 \\ 0 & 0 & 1.5 & 0.5 & -1 \\ -1 & -1 & 0.5 & 2.5 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

# MME

$$\begin{bmatrix} X^T X & X^T Z \\ Z^T X & Z^T Z + A^{-1} * \lambda \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{u} \end{bmatrix} = \begin{bmatrix} X^T y \\ Z^T y \end{bmatrix}$$

# Insert Data

$$\begin{bmatrix} 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 0 & -2 & 0 \\ 1 & 1 & 4 & 0 & -2 & 0 \\ 1 & 0 & 0 & 4 & 1 & -2 \\ 1 & -2 & -2 & 1 & 6 & -2 \\ 1 & 0 & 0 & -2 & -2 & 5 \end{bmatrix} \begin{bmatrix} \mu \\ \hat{u}1 \\ \hat{u}2 \\ \hat{u}3 \\ \hat{u}4 \\ \hat{u}5 \end{bmatrix} = \begin{bmatrix} 19.8 \\ 4.5 \\ 2.9 \\ 3.9 \\ 3.5 \\ 5 \end{bmatrix}$$

# Animal 4

- parents 1 and 2
- progeny 5
- mate 3
- inspection of second but last equation in MME where $y_4$ and $\hat{u}_4$ occur
- Remember from construction of $A^{-1}$, the variable $d^{ii}$ can assume the following values

$$d^{ii} = \begin{cases} 2 & \text{both parents known} \\ \frac{4}{3} & \text{one parent known} \\ 1 & \text{both parents unknown} \end{cases}$$

# Extract Equation

$$y_4 = 3.5 = 1 * \hat{\mu} - 2 * \hat{u}_1 - 2 * \hat{u}_2 + 1 * \hat{u}_3 + 6 * \hat{u}_4 - 2 * \hat{u_5}$$

- Solving for $\hat{u}_4$

$$\hat{u}_4 = \frac{1}{6}\left[y_4 - \hat{\mu} + 2 * (\hat{u}_1 + \hat{u}_2) - \hat{u}_3 + 2\hat{u}_5\right]$$

- $\hat{u}_4$ depends on
  - own performance record $y_4$
  - estimate of fixed effect $\hat{\mu}$ - environment
  - predicted breeding value of parents 1 and 2, mate 3 and progeny 5

# General Equation

$$\hat{u}_i = \frac{1}{1 + \alpha\delta^{(i)} + \frac{\alpha}{4}\sum_{j=1}^{n}\delta^{(k_j)}} \Bigg[ y_i - \hat{\mu} + \frac{\alpha}{2}\left\{\delta^{(i)}(\hat{u}_s + \hat{u}_d) + \sum_{j=1}^{n}\delta^{(k_j)}(\hat{u}_{k_j} - \frac{1}{2}\hat{u}_{l_j})\right\}\Bigg]$$

where

| | |
|---|---|
| $\alpha$ | ration between variance components $\sigma_e^2/\sigma_u^2$ |
| $\delta^{(j)}$ | contribution for animal $j$ to $A^{-1}$ |